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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository Mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

1. All manuscripts should be typewritten with double spacing and with margins at least one inch wide.

2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the MAGAZINE.

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NOBLESSE OBLIGE!

The solidarity of American Mathematical interests in the North, East, South and West have been beautifully evidenced in recent weeks. Our two-day visit to the Bethlehem Mathematical Meetings (National Council, M. A. of A., A. M. S.) was a gratifying revelation that behind NATIONAL MATHEMATICS MAGAZINE stand hosts of sympathetic well-wishers and supporters. A recent recommendation by the Louisiana State University Budget Committee that the budget of NATIONAL MATHEMATICS MAGAZINE for 1942-43 be materially cut served as a signal for prompt and earnest letters directed to President Hodges of this University—letters pleading for the continuation of an undiminished Magazine budget, pleading for a continuation of the same amount of service to the cause of mathematics that L.S.U. has so generously provided in the six years of its sponsorship of the Magazine.

We are confident that neither President Hodges nor those here named will object to our listing all whose letter-carbons have been sent to us. Final settlement of the Magazine budget is being deferred to the March meeting of the Board of Supervisors. Profound is our gratitude to the following:

The Mathematical Association of America (letter through Secretary Cairns and President Brink)
Mary Potter, President of National Council of Teachers of Mathematics
J. R. Kline, Secretary of American Mathematical Society
B. F. Finkel, Founder of American Mathematical Monthly
E. J. Moulton, former editor of American Mathematical Monthly
A. J. Kempner, Past President of Mathematical Association of America
C. H. Sisam, Professor of Mathematics, Colorado College
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R. E. Langer, Professor of Mathematics, University of Wisconsin and Vice-President of M. A. of A.
Caribel Kendall, Associate Professor, University of Colorado

S. T. SANDERS

The Trisection Problem

By ROBERT C. YATES
Louisiana State University

CHAPTER V

DON QUIXOTES

P. L. Wantzel in 1837 (see Liouville's Journal, II, p. 366) was the first to give a rigorous proof of the impossibility of trisecting the general angle by straightedge and compasses. (Gauss had already made the statement in his *Disquisitiones Arithmeticae* but neglected to give the proof). He was able to do this, however, only after far-reaching discoveries had been made in the fields of algebraic analysis and number theory. Since this date, other demonstrations by Klein [30] in 1895, Enriques [17] in 1900, Dickson [14] in 1914, etc., have appeared in more modern notation. Yet in the face of these conclusive proofs we still find a tremendous host advancing to the attack, armed only with straightedge and compasses. Some persistent stubbornness in our human race keeps this army at war-time strength and for each casualty there is at least one recruit ready to bear arms, indeed the same ones, in an effort to revise the scientific world and make it safe for the mathematician.

Once the virus of this fantastic disease gets into the brain, if proper antiseptics are not immediately applied, the victim begins a vicious circle that leads him from one outrage of logic to another. Consistently inconsistent, he slides under each fence, clears his conscience, and proceeds blithely to the next truth only to violate that in turn. It seems generally characteristic that all of these individuals have a superb command of flowery and bewitching language to tempt the uninitiated and gullible. To the professional mathematician, these phrases seem to serve but one purpose—to obscure the very violations that are always lurking in the proposals under one guise or another. These violations are at times very difficult to discover. But once brought to light, usually no amount of patient persuasion can convince the author of his error. Strangely enough, each new "solver" can

This is the last in a series of five chapters. The entire series is now available in book form.

see glaring mistakes in the work of his predecessor but is apparently oblivious to his own.

The fact that simple reasoning can accomplish nothing toward setting these people right has forced the professional mathematician to meet each proffered challenge in deep silence. The result, of course, in the already warped mind of the "savior of science" is the deep-rooted conviction that all mathematicians are in league against him. And that in itself becomes yet another unassailable argument of his infallibility. As a last resort, he turns to the layman through the medium of the daily newspapers much to the detriment of public faith in the professional mathematician.

Once a person has convinced himself that he has solved one of the Famous Problems that "have defied mathematicians for over two thousand years", it is but a short step to the realization that he is endowed with unusual powers. These powers are then focused upon all the other paradoxes from perpetual motion to the existence of God, and with characteristic consummate success. One professor of mathematics writes as follows:

"Quite often I receive letters from some individual who has discovered a kinship between phenomena which to the benighted scientist appear worlds apart. One, possessed by a truly universal spirit, has succeeded in uniting into a single synthesis the Euclidean postulate of parallels and the quadrature of the circle, the Fermat problem and perpetual motion, the principle of relativity and the existence of the Deity, the quantum theory of the atom and the forecasts of the stock market, the abolition of wars, the solution of the economic depression and the liberation of mankind from the Bolshevik scourge—to mention but a few of the achievements he claims."

A typical person of this sort was Mr. L. S. B—. His self assurance was so great that he offered a thousand dollars to the one who would prove wrong his argument in support of the value 3 for π .

A very recent "solver" of the Trisection Problem announced his discovery to the editor of an American mathematical journal but refused to disclose the nature of the solution until he had been awarded the sum of \$15,000. That amount, he said, was only the just compensation of an ordinary school teacher for services over the fifteen years that he had devoted to the problem—an amount that could very well come, he said, from football receipts.

CASE HISTORIES

1. *The Case of J. C. W—.*

In 1902 a little book, *Trisection of Angles* by Mr. W—, appeared with the explanatory preface:

"... It was necessary to get outside of the problem to solve it, and it was not solved by a study or knowledge of Geometry or Trigonometry, as the author had never made a study of these branches of learning. The proof was arranged in Geometrical order and formula by Ada S. Flood.

"The problem might have remained unsolved except for a study and analysis of the little poem, 'In the Distance', wherein the numbers 3 and 7 seem to coincide in various ways and wherein various other coincidences are demonstrated by the aid of progressive or triangular numbers. Herein was found the key to the solution of the problem:

"IN THE DISTANCE"

I

*The countless legions passed away,
And all the hosts on earth to-day,
Like vanished dreams may be forgot,
Their names and deeds remembered not,
Their gilded glories gone;
Their works as rust and desert dust,
Fame's phantom shadows flown.*

II

*Or like enchanted music rung,
Our songs attuned to cadence sung,
Or names by mystic fate renowned,
By glamour'd ancient glories crowned
With all that fame endears,
It nought would be to you or me,
Far down the distant years.*

III

*A few at most our troublous days;
Unto the vast unknown we gaze;
A glimmer of Immortal dawn,
A star of hope still shining on,
Gleams through the darkest sky;
A trust that good shall cross the flood,
And only evil die.*

IV

*Where doubt exists a hope may live;
None know the gifts that time may give;
Above our highest hopes and far
Beyong the dreamer's brightest star,
Have faith! for us may rise
The future's dawn, the shores unknown,
The fadeless Eden skies.*

V

*Let patience ever shield thy breast
From storm-tossed waves of wild unrest,
And love make all thy pathways bright,
Contentment make thy burdens light;
Let gloomy thoughts forlorn,
And griefs and fears, the pains and tears,
All pass like mists of morn.*

VI

*Haste not to leap the fabled stream;
What waits beyond we may not dream;
Rejoice to-day, yet meekly trust,
That only good above our dust,
By fate, somewhere, somehow,
From acts of ours may grow as flowers,
In far-off years from now.*

VII

*Trust now in fame now wealth to bless;
Go help the poor and soothe distress;
Be brave, be true and do your best;
Do good until with God you rest,
In some far wondrous home,
And all will be as well with thee,
Through all the years to come.*

"Coincidences"

"... There are as many syllables to the verse as there are weeks to the year, and 52 punctuation marks are used in the 7 verses. There are 365 syllables in the 7 verses. Also, the second and fourth verses combined have 365 letters,

and fourth and sixth verses combined have 365 letters, corresponding to the number of days in one year. . . . The first letter of the alphabet is used as a word and for the commencement of words 33 times; 33 commas are used; there are 33 letters in the longest line and 33 lines preceding it. There are 24 letters in the last line and 24 letters in the first word of each verse combined; the sum of all numbers from 1 to 24 = 300, the number of words in the 7 verses. . . . The number of letters in the alphabet, 26, multiplied by the number of verses, 7, = 182, the number of letters in the 7th verse. . . . The most wonderful of all numbers is 1287. The number of verses, 7, multiplied by the number of letters 1287 = 9009: the answer reads the same either way backward or forward. The sum of all numbers from 1 to 1287 = 828828 which reads same either way. . . . The sum of all numbers from 1 to 7 = 28. 'God' is the 28th word of the 7th verse in the 4th line, and the 279th word of the work. . . . The sum of all numbers from 1 to 10 = 55: 'Mystic' is the 55th word of the work in the 10th line. Commencing with the Sun as 1, Mercury as 2, Venus 3, Earth 4, Mars 5, The Asteroids 6, Jupiter 7, Saturn 8, Uranus 9, Neptune 10, Comets 11, the Fixed Stars and Nebula 12, and 13th the Unknown: 13 multiplied by the number of verses, $13 \times 7 = 91$. 'Unknown' is the 91st word of the work. . . ."

Although Mr. W— lays considerable stress upon the poem and its numerical oddities, he fails to reveal its connection with the Trisection Problem. The error in his solution is the assumption that a certain arc in the construction is circular. This arc, however, was shown to be hyperbolic by Pappus in the 3rd Century. We need not enter into the details of the construction here.

2. *The Case of J. W.—*

Mr. W—, B. A., M. D., Edin., a native of Greenock, went to considerable pains and expense to publish in 1911 a magnificent book of 169 pages called *The Trisection of the Angle by Plane Geometry*. In the preface he calms the reader by assuring him that he need only understand the geometry of Euclid in order to digest his treatment. Unfortunately, Mr. W— labored under the delusion that calculations carried out to seven place accuracy were sufficient proof of his method. The editor of the *Mathematical Gazette* reviewed this book as follows:

"Dr. W— has found a formula for the third part of a given angle, and applies it to fifty selected cases. . . . This stately marshalling of the arithmetical procedure is worthy of a better cause than the computation of sines and cosines to seven figures. . . . He seems to be quite aware of the fact that the problem has been classed among those that are insoluble, and quotes from De Morgan to that effect. We fear that he may continue to hug his comfortable delusion in spite of all that can be said. . . ."

3. *The Case of J. J. C.—*

Mr. C—, president of an American university, published in 1931 the two works: *Euclid or Einstein. A Proof of the Parallel Theory and a*

Critique of Metageometry; and The Trisection of the Angle. The Trigonometric Functions of One-third of an Angle in Terms of the Functions of the Angle. The Insertion of Two Geometric Means Between a line and Another twice as Long. The Duplication of the Cube. Et al.

The first is a book of more than 300 pages which gives emphasis to the author's opening sentence:

"We are surely living in a strange intellectual age."

In it Mr. C—"proves" the parallel postulate and concludes that the only geometry that can possibly exist is Euclidean. His attitude toward modern investigations is disclosed in the following quotations:

"... This age has gone further in this respect than any other; it has extended its attacks to the utmost bounds of science. The mutineers against the old order have seized the ship of knowledge and nailed the flag of dissent to the mast; they have driven the defenders of all manner of orthodoxy below decks and battened down the hatches over them, and have left in their administration not a single department of science. ... When normally sound criticism turns into destructive bolshevism, it is time to inquire whether the criticism is as sound as that which it criticises."

"As a result of this failure (to prove the parallel postulate), certain mathematicians of the last century came to the conclusion that the postulate was indemonstrable, certainly a very easy way to cut the Gordian knot of the difficulty; and then with the utmost in consequence, and with more mental agility than either poise or balance, jumped to the other and much more radical and subversive conclusion, that the proposition itself was not valid."

This last is a misstatement. Mathematicians did not conclude that the postulate was invalid; they simply replaced it with another one which is consistent with the rest and upon this foundation created the vast and important non-Euclidean geometry.

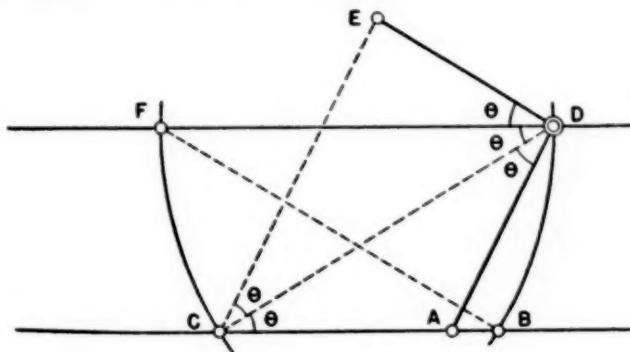


FIG. 40

The second work of Mr. C—disposes of the problem of Trisection. Because its absurdity is both simple and interesting we shall give the method here. The lines BC and DF are drawn parallel to each other. With any point, such as D , as center, describe the circular arc FC . With the same radius and center F draw the arc DB . Construct the angle DCE equal to angle DCA . Draw DA parallel to EC and DE parallel to FB . Then DF and DC trisect angle ADE .

Nothing could be truer or more fundamentally sound. However, Mr. C—has his cart before the horse. Instead of trisecting a given angle, he has erected from an arbitrarily chosen angle DCA its *triple*, angle ADE . Due to the respected position that he held in the educational world, Mr. C—unfortunately received considerable notoriety for this bit of mathematical play. The newspapers of the day made much of his “discovery” and undoubtedly created much excitement in the ranks of the layman.

A curious paragraph in the same pamphlet lists the trigonometric functions of one-third of an angle in terms of the angle:

$$\begin{array}{ll} \sin(A/3) = 2 \sin A + \tan A; & \cos(A/3) = 2 \sec A + 1; \\ \tan(A/3) = 2 \sin A; & \sec(A/3) = 2 \cos A + 1; \\ \cot(A/3) = 2 \csc A; & \csc(A/3) = 2 \sin A + \cot A. \end{array}$$

Using these formulas to calculate the functions of 30° , letting $A = 90^\circ$, we find:

$$\begin{array}{lll} \sin 30^\circ = \infty & \cos 30^\circ = \infty & \tan 30^\circ = 2 \\ \sec 30^\circ = 1 & \csc 30^\circ = 2 & \cot 30^\circ = 2. \end{array}$$

This display seems doubly strange when we remember that the square of the sine added to the square of the cosine equals 1.

4. *The Case of J. J. G.*

Mr. G—, onetime instructor of mathematics in a college of California, published in 1932 a beautiful little book under the title: *The Mathematical Atom*. That it struck a popular note among the interested public is evidenced by the fact that three editions appeared in scarcely more than a year's time. He recounts the “success” of his struggles with the Trisection Problem:

“In the course of the attempt and upon closer scrutiny I found the *two lines* mathematicians had been in search of since the days of old Pythagoras securely linked up with a couple of sets of parallel lines crisscrossing each other and together forming ‘perspectives of pleasant shades and wide open spaces’;

and the two *distinguished points* nestling in the heart of two mutually overlapping right triangles, perched upon two tangents to a circle at the ends of two of its radii; and the *three great points* *O* and *A* and *B* dominating the whole expanse of the angle's empyrean."

Farther on, Mr. G— tells of his discovery of a new kind of triangle that seems to him destined to play a vital role on the mathematical stage:

"The *Golden Mean Triangle* will serve to show that even *the scalene triangle* is not to be classed among the lower host of things, 'the loose, the lawless, the exaggerated, the insolent, and the profane'. For though the scalene triangle may appear at a first glance to be something of a *sans-culotte*, and sartorially and æsthetically not quite on a par with the more aristocratic triangles, the capricious little vagabond can nevertheless be shown fundamentally and potentially to possess the properties of beauty and symmetry, even as it possesses the other metaphysical properties of truth and goodness,—which things are ontologically inherent in all of God's creations, yea, in their every tiniest atom or fragment, however humble or commonplace."

We need not comment upon these passages. It is regrettable that lack of space forces us to reject Mr. G—'s invitation to an excursion:

"... if you want to take a jaunt out into the belt of any angle, wide, narrow, or straight, and want to make equally good and spacious reservations for yourself and two companions, hitch your wagon to the twin stars—ALPHA and BETA GEMINORIUM; give them the reins, and they'll take you to see half a hundred points of interest on a tour through their vast domain, including a number of delightful stopovers at their own commanding coignes of vantage, leaving you—heart and fancy free—to walk and ramble about in the garden of the manor, *dolce far niente*, to your heart's content. Or to pause and invite your soul to rest... the while you hearken to the distant cosmic harmonies of the whirling spheres, as their echoes come crashing upon the treetops in diapasonic over- and under-tones, running through all the compass of the notes, symphonically blending with the rustling music of the forests, strummed out by fairy fingers upon a thousand harps of sunbeams piercing the fragrant shadows of the giant primeval groves."

Toward the latter part of the book we find Mr. G—'s method of "trisection": Describe the arc *AB* upon the given angle *AOB*. Draw lines *OC* bisecting the angle; and *OD* bisecting the half. The tangent to *AB* at *E* intersects *OC* at *F*. Draw *FG* parallel to *OD*. With *F* as center and *FO* as radius describe the arc cutting out the points *D*, *G*, *C*. Draw *FD* and *EG* which intersect each other at *X*. Then *OX* is the "trisecting" line.

We leave to the reader the fun of spotting the error in this method. As an approximation it is excellent.

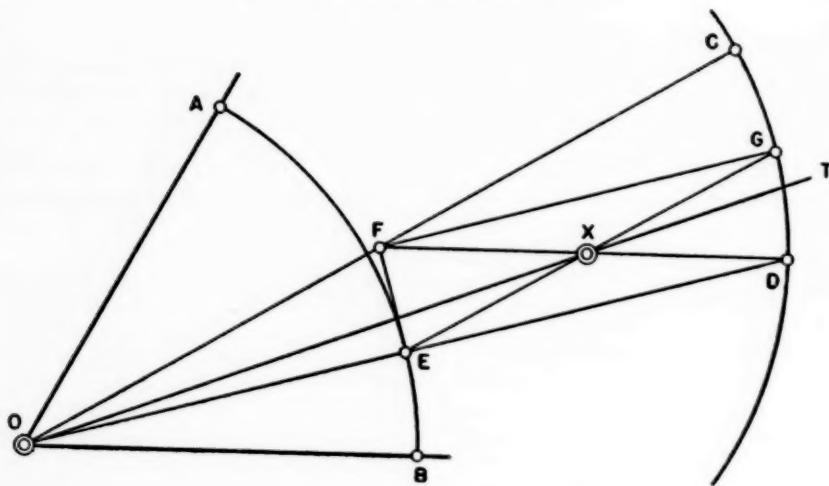


FIG. 41

5. *The Case of L. J. R. H.—*

It is an infrequent occurrence that a purported straightedge and compasses "trisection" should appear in a serious periodical devoted to science. Through the editorial offices of every journal there passes a continual stream of new "solutions" which are either returned promptly to the authors or just as promptly consigned to rightful oblivion in the waste basket. Although every editor is constantly on guard, some of these attempts do slip through to the printed page. An instance of this is to be found in the paper: "A Solution for the Geometrical Trisection of Angles and the Proportional Dividing of Arcs" by L. J. R. H.—. Mr. H— there gives two methods of "trisection", one of which follows:

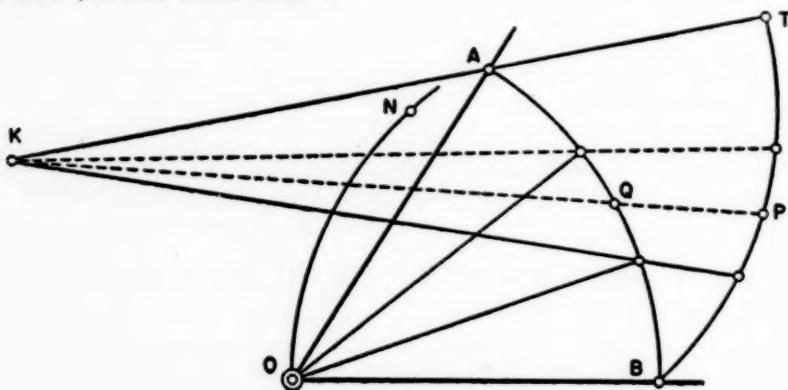


FIG. 42

Given the angle AOB . Draw an arc BA with center O . With the same radius and center B draw arc ON . With center at an arbitrary point N on ON , same radius, draw arc BT of any length and divide it into three equal arcs. Bisect arcs BA , BT to obtain the points P and Q . Draw the lines TA and PQ which intersect at K . Then lines drawn from K to the trisecting points of the arc BT "trisect" the arc BA .

It is easy to show that a fallacy exists and that the length of the arc BT and the position of the point N cannot be chosen at random. Thus, for example, if BT be taken as a semicircle the construction will yield the following "trisection" for 60° :

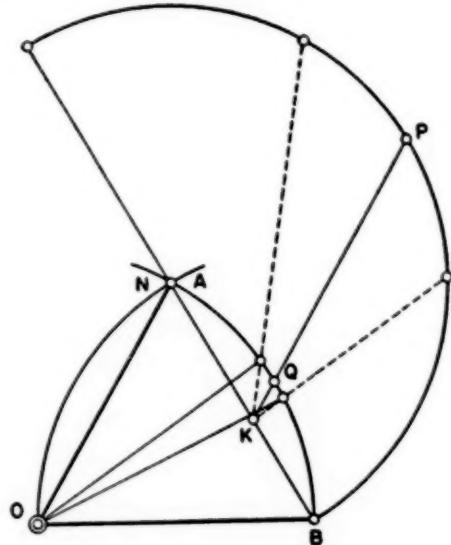


FIG. 43

Since this does not even appeal to the eye, the method obviously does not work.

6. Miscellaneous Cases

1. *The Trisection of the Angle* by J. A. L—, (1890) ("being a problem in Geometry that has baffled the efforts of mathematicians up to the present day, now solved for the first time.")
2. *The Geometrical Problem Solved* by H. D. D—, (1892).
3. *Geometrical Division and Measurement of Arcs and Angles* by N. J—, (1900) ("the first person in the world to trisect, penta-

sect, and hepta-sect arcs and angles geometrically, or to measure arcs and angles without compass or protractor.").

4. *A New Method of Trisecting Any Angle and of Constructing a Regular Pentagon with Ruler and Compasses* by H. A. E—, Supt. of City Schools, Slater, Mo. (no date).
5. *Trisectio Arcus et Anguli* by J. W. Th. O—, (1906) ("... and hereby we give to the world the solution of this remarkable problem of twenty odd centuries. May the tired spirits of the past from Pythagoras and Euclid to Newton now rest in peace! We are happy ourselves at last to feel entitled to rest.")
6. *The Trisection of an Angle* by J. S—, (1914).
7. *Trisecting an Angle by Compass and Straightedge* by E. H. Y—, (1931).
8. *Euclidean Trisection, Quintisection, and Hexasection* by A. A. Z—, (1932).
9. *Solution of an Insolvable Problem* by B. D. H—, (1932).
10. *The Trisection of the Angle and Theorems and Corollaries Leading To It (Revised)* by F. S—, (1933).
11. *Trisecting an Angle of any General Magnitude* by L. A. McC—, (1934).
12. *Youth Claims Formula Great Mathematicians Seek*, Associated Press Dispatch, (Aug. 31, 1935).

BIBLIOGRAPHY

1. Adler, A.: *Theorie der geometrischen Konstruktionen*, Leipzig, 1906.
2. Amadori: *Trisezione d'un angolo qualunque mediante riga e compasso*, Savona, 1883.
3. Aubry: *Journal de Mathématiques Spéciales*, 1896, pp. 76-84; pp. 106-112.
4. Ball, W. W. R.: *Mathematical Recreations and Essays*, London, 1940.
5. Breidenbach, W.: *Die Dreiteilung des Winkels*, Leipzig, 1933.
6. Brennan, M. H.: *The Trisection of the Arc*, Devil's Lake, Dakota, 1888.
7. Brocard, H.: (a) *Note sur la division mécanique de l'angle*, Bull. Soc. Math. de France, III, 1875, pp. 47-48; V, 1876, pp. 43-47.
(b) *Notes de Bibliographie des Courbes Géométriques*, 1897, p. 290.
8. Cajori, F.: *A History of Mathematics*, New York, 1926.
9. Ceva, Th.: *Acta Erud.*, MDCXCV (1695), p. 290.
10. Dantzig, T.: *Number, The Language of Science*, New York, 1930.
11. De Morgan, A.: *A Budget of Paradoxes*, London, 1872.
12. Descartes, R.: *La Géométrie*, Paris, 1637, Berlin, 1894.
13. Dexter, O. P.: *The Division of Angles*, American News Co., New York, 1881.

14. Dickson, L. E.: Elementary Theory of Equations, New York, 1914.
15. Dürer, A.: Unterweysung der messung mit dem zirkel und richtscheit, Nürnberg, 1525.
16. Edington, E. E.: House Bill No. 246, Indiana State Legislature, 1897, Proc. Indiana Academy of Science, 45, 1936, pp. 206-210.
17. Enriques, F.: Fragen der Elementargeometrie, II, Leipzig, 1911.
18. Ferguson, D. F.: Geometrical Construction for the Trisection of an Angle to any Required Degree of Accuracy, Mathematical Gazette, 9, 1919, p. 373.
19. Fialkowski: Teilung des Winkels und des Kreises, 1860, p. 11.
20. Genese, R. W.: On the Trisection of an Angle, Messenger of Mathematics, I, 1872, pp. 103, 181.
21. Givens, W. B.: The Division of Angles into Equal Parts and Polygon Construction, American Mathematical Monthly, XLV, 1938, pp. 653-656.
22. Good, A.: Scientific Amusements, (no date: reprinted about 1937).
23. Heath, T. L.: Greek Mathematics, Oxford, 1921.
24. Hilbert, D.: The Foundations of Geometry, Chicago, 1902.
25. Hippauf, H.: Lösung des Problems der Trisection, Leipzig, 1872.
26. Hudson, H. P.: Ruler and Compasses, London, 1916.
27. Hutton: Philosophical Recreations, 1844.
28. Juredini, G. M.: A New Curve Connected with Two Classical Problems, American Mathematical Monthly, 33, 1926, p. 377 ff.
29. Kempe, A. B.: (a) Messenger of Mathematics, NS IV, 1875, pp. 121-124.
(b) How to Draw a Straight Line, New York, 1877.
30. Klein, F.: Famous Problems of Elementary Geometry, Boston, 1897.
31. Kortum, H.: Über geometrische Aufgaben dritten und vierten Grades, Bonn, 1869.
32. Lagarrique, J. F.: The Trisection Compass, New York, 1831.
33. Laisant, C. A.: Note sur un Compas Trisection, Compte rendu (Congrès de Nantes), 1875, pp. 61-63.
34. Lucy, A. W.: (a) A Method of Trisecting an Angle, Mathematical Gazette, 11, 1922, p. 21.
(b) To Divide an Angle into any Number of Equal Parts, Mathematical Gazette, 14, 1928, pp. 137-138.
35. Mitzscherling, A.: Das Problem der Kreisteilung, Leipzig, 1913.
36. Montucla: Histoire des recherches sur la Quadrature du Cercle, Paris, 1831.
37. Nicolson, T. W.: The Multisection of Angles, The Analyst, X, 1883, pp. 41-43.
38. Ocagne, M. d': Etude rationnelle du Problème de la Trisection de l'Angle, L'Enseignement Math., 1934, pp. 49-63.
39. Pappus: Coll. Math., Prob. VIII, Prop. XXXII.
40. Perron, O.: Sitzungsberichte der Bayerischen Academie der Wissenschaften, 1933, pp. 439 ff., 1929, pp. 341 ff.
41. Priestley, H. J.: Duplication, Trisection and Elliptical Compasses, Mathematical Gazette, 12, 1924, pp. 212-216.
42. Schubert, H. C. H.: Mathematical Essays and Recreations, Chicago, 1917.
43. Scientific American, July, 1933; April, 1936, pp. 190-191; 228-229.

44. Scudder, H. T.: How to Trisect an Angle with a Carpenter's Square, American Mathematical Monthly, 1928, pp. 250-251.
45. Sidler, G.: Zur Dreiteilung eines Kreisbogens, Bern, 1876.
46. Smith, J. C.: Ann. di Mat., 3, 1869, p. 112; Collected Math. papers 2, p. 1.
47. Sylvester, J. J.: Collected Works.
48. Tropfke: Geschichte der Elementar-Mathematik, Leipzig, 1930
49. Vahlen, T.: Konstruktionen und Approximationen, Leipzig, 1911
50. Wulff-Parchim, L.: Dürer als Mathematiker, 1928.
51. Yates, R. C.: (a) A Trisector, NATIONAL MATHEMATICS MAGAZINE, XII, 1938, pp. 323-324.
 (b) Line Motion and Trisection, Ibid., XIII, 1938, pp. 1-4.
 (c) The Angle Ruler, the Marked Ruler, and the Carpenters' Square, Ibid., XV, 1940, pp. 61-73.
52. For a general bibliography, see the supplements to L'Intermédiaire des Mathématiciens, Paris, May and June, 1904.

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The Standard Error of the Standard Error of Estimate

By WILLIAM DOWELL BATEN
Michigan State College

When a scatter diagram pertaining to n pairs of observed values of two variables lies along a straight line

$$(1) \quad y = a + b(x - \bar{x}),$$

the standard error of estimate, for large n , is the square root of the sum of the squares of the deviations of the values of the dependent variable from the predicting line (1) divided by n ; or is

$$(2) \quad \sigma_e = \sqrt{\frac{\sum (y_i - Y')^2}{n}},$$

where Y' is the value obtained from the predicting equation and y_i the observed value. The standard error of estimate can be written as

$$(3) \quad \sigma_e = \sigma_y \sqrt{1 - r_{xy}^2} = \bar{p}_{02} \sqrt{1 - r_{xy}^2} \quad \text{or}$$

$$(4) \quad \log \sigma_e = \frac{1}{2} \log \bar{p}_{02} + \frac{1}{2} \log (1 - r^2),$$

where $\bar{p}_{ij} = \frac{\sum (x - \bar{x})^i (y - \bar{y})^j}{n}$.

Taking the derivative of both members of (4) gives

$$(5) \quad \frac{\delta \sigma_e}{\sigma_e} = \frac{\delta \bar{p}_{02}}{2\bar{p}_{02}} - \frac{r \delta r}{1 - r^2},$$

where $\delta \sigma_e$ represents an error in σ_e , $\delta \bar{p}_{02}$ an error in \bar{p}_{02} and δr an error in r_{xy} due to fluctuations in sampling.* Squaring both members of (5) and averaging for all possible samples gives

$$(6) \quad \frac{\sum (\delta \sigma_e)^2}{S \sigma_e^2} = \frac{\sum (\delta \bar{p}_{02})^2}{4 \cdot S \cdot \bar{p}_{02}^2} - \frac{r \sum (\delta \bar{p}_{02})(\delta r)}{S \cdot \bar{p}_{02} \cdot (1 - r^2)} + \frac{r^2 \sum (\delta r)^2}{S \cdot (1 - r^2)^2},$$

where S represents the number of possible samples.

* "On the probable errors of frequency constants Part II Editorial" Bio. Vol. IX, 1913, pp. 1-10.

From the definition of the correlation it is found that an error in the correlation coefficient is

$$(7) \quad \delta r = r_{xy} \cdot \left[\frac{\delta \bar{p}_{11}}{\bar{p}_{11}} - \frac{1}{2} \frac{\delta \bar{p}_{20}}{\bar{p}_{20}} - \frac{1}{2} \frac{\delta \bar{p}_{02}}{\bar{p}_{02}} \right]^*$$

Multiplying both members of (7) by $\delta \bar{p}_{02}$ and averaging for all samples gives

$$\frac{\Sigma(\delta \bar{p}_{02})(\delta r)}{S} = r \cdot \left[\frac{\bar{p}_{13} - \bar{p}_{02}\bar{p}_{11}}{n\bar{p}_{11}} - \frac{1}{2} \frac{\bar{p}_{22} - \bar{p}_{20}\bar{p}_{02}}{n\bar{p}_{20}} - \frac{1}{2} \frac{\bar{p}_{04} - \bar{p}_{02}^2}{n\bar{p}_{02}} \right].$$

Substituting this value of $\frac{\Sigma(\delta \bar{p}_{02})(\delta r)}{S}$ in (6) gives

$$(8) \quad \frac{(\sigma_{\sigma_r})^2}{\sigma_r^2} = \frac{1}{4} \frac{\bar{p}_{04} - \bar{p}_{02}^2}{n\bar{p}_{02}^2} - \frac{r^2}{n\bar{p}_{02}(1-r^2)} \left[\frac{\bar{p}_{13} - \bar{p}_{02}\bar{p}_{11}}{\bar{p}_{11}} \right. \\ \left. - \frac{1}{2} \frac{\bar{p}_{22} - \bar{p}_{20}\bar{p}_{02}}{\bar{p}_{20}} - \frac{1}{2} \frac{\bar{p}_{04} - \bar{p}_{02}^2}{\bar{p}_{02}} \right] + \frac{r^2 \sigma_r^2}{(1-r^2)^2}.$$

If it is assumed that the regression of y on x is linear and also that the regression of x on y is linear, we may put

$$\frac{\bar{p}_{13}}{\bar{p}_{11}} = \sigma_y^2 \beta'_2, \quad \frac{\bar{p}_{04}}{\bar{p}_{02}} = \sigma_y^2 \beta'_2, \quad \text{where } \beta'_2 = \frac{\Sigma(y - \bar{y})^4}{N\sigma_y^4}.$$

Pearson† proved that

$$\frac{\bar{p}_{22}}{\bar{p}_{02}} - \bar{p}_{20} = \sigma_x^2 r_{xy}^2 \sqrt{(\beta_2 - 1)(\beta_2' - 1)},$$

approximately, where $\beta_2 = \frac{\Sigma(x - \bar{x})^4}{N\sigma_x^4}$.

Pearson showed also that*

$$\sigma_r = \frac{1-r^2}{\sqrt{n}}.$$

* "On the probable errors of frequency constants Part II Editorial" Bio. Vol. IX, 1913, pp. 1-10.

†(2), (3) *Loc. cit.*

Substituting these values in (8) gives

$$\frac{(\sigma_{\sigma_e})^2}{\sigma_e^2} = \frac{\frac{1}{4}\beta_2' - \frac{1}{4}}{n} - \frac{r^2}{n(1-r^2)} [\beta_2' - 1 - \frac{1}{2}r^2\sqrt{(\beta_2-1)(\beta_2'-1)} - \frac{1}{2}\beta_2' + \frac{1}{2}] + \frac{r^2(1-r^2)^2}{n(1-r^2)^2} \quad \text{or}$$

$$(9) \quad \sigma_{\sigma_e} = \sigma_e \sqrt{\frac{\frac{1}{4}\left(\frac{\beta_2'-1}{n}\right) - \frac{r^2}{2(1-r^2)} \left[\frac{(\beta_2'-1) - r^2\sqrt{(\beta_2-1)(\beta_2'-1)}}{n} \right] + \frac{r^2}{n}}{n}}$$

If the distribution of the x 's and y 's are such that their kurtoses are equal to zero the above becomes

$$(10) \quad \sigma_{\sigma_e} = \sigma_e \sqrt{\frac{1}{2n} - \frac{r^2}{2(1-r^2)} \left[\frac{2-2r^2}{n} \right] + \frac{r^2}{n}} \quad \text{or}$$

$$\sigma_{\sigma_e} = \sigma_e \sqrt{\frac{1}{2n} - \frac{r^2}{n} + \frac{r^2}{n}} = \frac{\sigma_e}{\sqrt{2n}}$$

which shows that the standard error of the standard error of estimate is similar to the standard error of a standard deviation.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

ÉMILE PICARD

According to a Reuters dispatch reported from Vichy, Professor Émile Picard, 85, internationally known mathematician and perpetual secretary of the French Academy of Sciences, died December 12 in Paris. Professor Picard was trained at the Sorbonne under Hermite and later became his son-in-law. He once said of Hermite: "Never was a professor less didactic, but never was a professor more lively." Following the fall of France in June, 1940, Professor Picard was compelled, despite his advanced age, by harsh vicissitudes to emigrate to Trag, Arabia. Even then he was deeply concerned with questions in the mathematico-physical sciences, as well as with problems and methods of the history, critique, and philosophy (taken together) of these sciences. He was forced to leave his papers and library in Paris, but hoped to carry on his research and publication. He gave a series of lectures on "Great Mathematicians" at the "Maison Carrée" in the Polytechnic University (Asia). Following World War I Professor Picard took the precautionary measure of establishing bank accounts outside France.

Picard's theorem was enunciated in 1879; his theory of algebraic space curves appeared in 1880, involving the problem of enumerating all space curves of a given order. He was also known for work on the general problem of birational transformation of surfaces by transcendental means.

Writing in 1902 of Hermite's scientific work, Picard said: "Hermite believed that numbers form (having their own existence outside us) a world of which we can know only in this existence some of the profound harmonies. In antiquity he would have been a Platonist, and in the

Middle Ages in the long quarrel between realism and nominalism he would have followed Guillaume de Champeaux with the realists."

Picard gave an interesting bit of mathematical philosophy in a memoir on the life and work of Georges-Henri Halphen (1890): "It seems that one could distinguish today two different tendencies of the mind, among mathematicians. Some busy themselves principally with enlarging the field of known ideas; without always exhausting the difficulties that they leave behind them, that do not fear to go ahead and investigate new subjects of study. The others prefer to remain in the domain of ideas already better developed in order to investigate longer; they desire to exhaust the consequences and cause themselves to take as evidence in the solution of each question the true elements on which it depends. These two directions of mathematical thought are observed in the different branches of the science; one can say always, in a general manner, that the first tendency is met more often in works which touch on integral calculus and the theory of functions; works on modern algebra and analytic geometry are raised especially by the second. It is to this that the work of Halphen is principally connected: this profound mathematician was above all an algebraist."

G. WALDO DUNNINGTON.

A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from November, 1941, issue)

THE AMERICAN JOURNAL OF MATHEMATICS

We are now coming to the consideration of some *American Mathematical Journals* of stellar magnitudes, and yet the two last considered may have shed mathematical light to a larger clientele perhaps than did those we are about to consider.

Following the *Analyst* in chronological order, the next mathematical journal published in the United States was the *American Journal of Mathematics*.

The contents of its front cover reads as follows:

AMERICAN
JOURNAL OF MATHEMATICS
PURE AND APPLIED

Editor in Chief,

J. J. SYLVESTER, LL.D., F. R. S., *Corr. Mem. Inst. of France.*

Associate Editor in Charge,

WILLIAM E. STORY, Ph.D., (*Leipsic.*)

With the Co-operation of

Benjamin Peirce, LL.D., F. R. S., Simon Newcomb, LL.D., F. R. S.,
Professor of Mathematics in Corr. Mem. Inst. of France,

Harvard University, Superintendent of the American
IN MECHANICS, Ephemeris,

IN ASTRONOMY,

and

H. A. Rowland, C. E.,
IN PHYSICS.

Published Under the Auspices of the
JOHNS HOPKINS UNIVERSITY.

Volume I.

Number 1.

BALTIMORE:

Printed for the Editors by John Murphy & Co.

B. Westermann & Co., *New York*. A. Williams & Co., *Boston*.
Ferree & Co., *Philadelphia*. Trübner & Co., *London*.

1878.

NOTICE TO THE READER.

In presenting to the Public this first number of the *American Journal of Mathematics*, the Editors think it advisable, in order to prevent disappointment on the part of subscribers and contributors, to state briefly the principles by which they will be guided in its management.

Although in the first instance designed to supply a want, as a medium of communication between American mathematicians, its pages will always be open to contributions from abroad, and promises of support from various foreign mathematicians of eminence have already been received.

The publication of original investigations is the primary object of the *Journal*. In addition to this, from time to time concise abstracts will be inserted of subjects to which special interest may attach, or which have been developed in memoirs difficult of access to American students.

Critical and bibliographical notices and reviews of the most important recent mathematical publications, American and foreign, will also form part of the plan. It is believed that many are prevented from ordering mathematical books from abroad by the uncertainty of ascertaining beforehand their real character, and thus are liable to be deprived of what might have been valuable aids to them in scientific investigation. This difficulty the editors hope to some extent to remove.

The *Journal* will be published in volumes of about 384 quarto pages, each volume appearing in four numbers at periods not absolutely fixed, but for the present, separated as nearly as may be found practicable by intervals of three months. In order that the numbers may not differ too much in size, an article will be made, when necessary, to run on from one number to the next, as is the practice with foreign journals of a similar nature.

This is the only journal of the kind in the English language published in the quarto form, the advantages of which to reader and author of mathematical papers are too well understood to need enumeration.

The editors believe it will materially aid in fostering the study of Mathematical Science throughout this continent and they feel it their duty to state that any good which may arise from it will be in a great measure due to the enlightened liberality of the Trustees of the Johns Hopkins University who have prompted the undertaking and guaranteed a considerable portion of the pecuniary risk attendant upon it.

It is to be understood that there will be no problem department in the *Journal*, but important remarks, however brief, may be inserted as notes. Persons desirous of offering to the Public mathematical problems for solution, are recommended to send them to

"*The Analyst*," edited and published by J. E. Hendricks, *Des Moines, Iowa*; or to

"*The Mathematical Visitor*," edited and published by Artemas Martin, *Erie, Pa.*

The subscription price of the *Journal* is \$5.00 a volume, and single numbers may be obtained at \$1.50.

Communications and subscriptions (by postal money order) may be addressed to

WILLIAM E. STORY,
Johns Hopkins University, Baltimore, Md.

Subscriptions will also be received by

B. Westermann & Co., *New York*.
A. Williams & Co., *Boston*.
Ferree & Co., *Philadelphia*.
Trübner & Co., *London*.

To give an idea of the class of persons whom it is expected the Journal will reach, the following list of the first 100 subscribers is inserted.

Of the 100 subscribers listed, we shall add only a few of the more familiar names

(F.)

M. Charles Hermite, Paris, France.
J. W. L. Glaisher, Cambridge, England.
C. A. Young, Bradford, Mass.
H. A. Newton, New Haven, Conn.
Elias Loomis, New Haven, Conn.
Miss Christine Ladd, Union Springs, N. Y.
De Volson Wood, Hoboken, N. J.
H. T. Eddy, Cincinnati, Ohio.
Edward Olney, Ann Arbor, Mich.
W. W. Beman, Ann Arbor, Mich.
Thomas Craig, Baltimore, Md.
Marcus Baker, Washington, D. C.
C. S. Peirce, New York.

This journal is still published (January, 1942) under the auspices of the Johns Hopkins University. There is an Index Number of 31 quarto pages to Volumes I-X. We shall give in full the Preface to this Index Number. It says:

"*The American Journal of Mathematics* was founded at the Johns Hopkins University in 1878, and was conducted, until 1884, by Professor Sylvester, assisted at different periods by Dr. William E. Story, Dr. Fabian Franklin, and Dr. Thomas Craig. In 1884, at the beginning of Volume VII, when Professor Sylvester left the University, the *Journal* came under the present management, Professor Simon Newcomb, Editor, and Dr. Thomas Craig, Associate Editor.

The *Journal* has among its regular contributors most of the leading mathematicians of the world, nearly every volume containing memoirs from eminent mathematicians in the United States, England, France, and Germany. Up to the end of Volume X, papers were published from eighty-nine contributors. This number of contributors is somewhat increased in volumes XI and XII, the former of which has already appeared, while much of the latter is now in progress."

Since this *Journal* has been published continuously from its beginning and is quite accessible, being in the Libraries of most of the Universities and the larger Colleges, we shall not incorporate many of the Titles of the Articles treated in the *American Journal of Mathematics*.

When the covers of each of the four numbers constituting the various volumes of the *American Journal of Mathematics* are removed and destroyed as is generally done when such journals are prepared for binding in book form, not a vestige of the history of its founding nor the names of its various noted Editors can be found within its covers. As times goes on, these items will be, comparatively, as difficult to find as the "Time Capsule" buried in the New York World's Fair Grounds, September 23, 1940, for the Earth's Inhabitants five thousand years hence, were nearly all the present inhabitants together with the products of their civilization completely destroyed, before that remote date.

We shall give the names of the Editors from Volume I to Volume XXXVIII inclusive. These names and the dates of publication are printed on the outside of the front covers of the four numbers of each volume.

The front covers of each of the four numbers of Volume I are the same as that on page 1, excepting each has its respective number. The cover of No. 2, Vol. I has added two companies who may receive subscriptions: D. Van Nostrand, *New York*, and A. Asher & Co., *Berlin*. The front cover of Vol. II, No. 1, has the following:

AMERICAN
JOURNAL OF MATHEMATICS
PURE AND APPLIED.

Editor in Chief: J. J. Sylvester,
Associate Editor in charge: William E. Story.
With the Co-operation of
Simon Newcomb, H. A. Newton, and H. A. Rowland.
Published under the Auspices of the
JOHNS HOPKINS UNIVERSITY.

Volume II.

Number 1.

BALTIMORE:

Printed for the Editors by John Murphy & Co.

B. Westermann & Co., *New York*. Trübner & Co., *London*.
D. Van Nostrand, *New York*. Gauthier-Villars, *Paris*.
Ferree and Co., *Philadelphia*. A. Asher & Co., *Berlin*.

March, 1879.

The front covers of the numbers of Vol. III, read as follows:

AMERICAN
JOURNAL OF MATHEMATICS

Editor in Chief: J. J. Sylvester.
Associate Editor in charge: William E. Story.
Published under the Auspices of the
JOHNS HOPKINS UNIVERSITY

Volume III.

Number 1.

Cambridge, University Press:

Printed for the Editors by John Wilson & Son.

B. Westermann & Co., *New York*. Trübner & Co., *London*.
D. Van Nostrand, *New York*. Gauthier-Villars, *Paris*.
Ferree & Co., *Philadelphia*. A. Asher & Co., *Berlin*.

Vols. IV and V were edited alone by J. J. Sylvester, Vol. VI was edited by J. J. Sylvester, Thomas Craig, Ph.D., Assistant Editor. Vols. (VII-XVI) inclusive, were edited by Simon Newcomb, Editor, Thomas Craig, Associate Editor. Beginning with Vol. XVII, the

numbers of the Volumes were published during the months of January, April, July, and October. Vols. (XVII-XXI) inclusive were edited by Thomas Craig with the cooperation of Simon Newcomb; Vol. XXII was edited by Simon Newcomb with the cooperation of A. Cohen, Frank Morley, Charlotte A. Scott, and other mathematicians. Vols. (XXIII-XXXI) inclusive were edited by Frank Morley with the cooperation of Simon Newcomb, A. Cohen, Charlotte A. Scott, and other mathematicians. Vols. (XXXII-XXXVIII), were edited by Frank Morley with the cooperation of A. Cohen, Charlotte A. Scott, and other mathematicians.

The writer's file of this journal is to and including No. 2, Vol. XXXVIII.

The *Journal* was founded in 1878. The month in which the first number of the first volume appeared is not certain. However, the only place where dates of months appear is in No. 2, Vol. I and No. 3, Vol. I, where the dates of July 1, and Sept. 1 are given in connection with the Editors' statement in the fifth paragraph of their Notice to the Reader, would lead one to infer that the first No. of Vol. I, appeared in January.

Contents of Vol. I, No. 1.

Note on a class of Transformations which Surfaces may undergo in space of more than Three Dimensions, by Simon Newcomb, pp. 1-4; Researches in the Lunar Theory, by G. W. Hill, Nyack Turnpike, N. Y.; pp. 5-26, The Theorem of Three Moments, by Henry T. Eddy, University of Cincinnati, pp. 27-31; Solution of the Irreducible Case, by Guido Weichold, of Zittan, Saxony, pp. 32-49; Desiderata and Suggestions, by Professor Cayley, Cambridge, England. No. 1.—The Theory of Groups, pp. 50-52; Note on the Theory of Electric Absorption, by H. A. Rowland, pp. 53-58; Esposizione del Metodo dei Minimi Quadrati. Per Annibale Ferrero, by Charles S. Peirce, New York, pp. 59-63; On an Application of the New Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics with three Appendices, by J. J. Sylvester, pp. 64-90; Appendix 1. Remarks on Differentiation Expressed in Terms of the Differences of the Roots of their Parent Quantics, by J. J. Sylvester, pp. 83-90; Appendix 2. Note on Hermite's Law of Reciprocity by J. J. Sylvester, pp. 90-104.

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On Hermite's Law of Reciprocity. Note A to Appendix 2. Completion to the Theory of Principal Forms. Note B to Appendix 2.

Additional Illustrations of the Law of Reciprocity, pp. 107-112. Note C to Appendix 2. On the Principal Forms of the General Sextinvariant to a Quartic and Quartinvariant to a Sextic, pp. 112-114. Note D to Appendix 2. On the Probable Relation of the Skew Invariants of the Binary Quantics and Sextics to one another and to the Skew Invariants of the same Weight of the Binary Nonic, pp. 114-118. Appendix 3. On the Clebsch's Theory of the "Einfachstes System Associerter Formen" (vide Binaren Formen, p. 330) and its Generalization. By J. J. Sylvester, pp. 118-125.

Extract of a letter to Mr. Sylvester from Professor Clifford of University College, London. pp. 126-128. Researches in the Lunar Theory. By G. W. Hill (*Continued from p. 26*) Chapter II, pp. 129-147. Bipunctual Coordinates. By F. Franklin, pp. 148-173. Desiderata and Suggestions. By Professor Calley. No. 2.—The Theory of Groups: Graphical Representation, pp. 174-176. On the Elastic Potential of Crystals. By William E. Story, pp. 177-183. Théorie des Fonctions Numeriques Simplement Periodiques. Par Edouard Lucas. pp. 184-196, continued to page 240, Vol. I, No. 3.

Contents of Vol. I, No. 3.

The Elastic Arch. By Henry T. Eddy, pp. 241-244. Researches in the Lunar Theory (*Continued from p. 147*). By G. W. Hill, pp. 245-260. Bibliography of Hyper-space and Non-Euclidean Geometry. By George Bruce Halsted, pp. 261-276. Some Remarks on a Passage in Professor Sylvester's Paper as to the Atomic Theory. *Contained in a letter addressed to the Editor by Professor I. W. Mallet of the University of Virginia.* pp. 277-281. Notes.—I. Historical Data concerning the Discovery of the Laws of Valence, p. 282; II. On the Mechanical Description of the Cartesian, p. 283. By J. H. Hammond. III. A New Solution of Biquadratic Equations. By T. S. E. Dixon, pp. 283-284. IV. On a Short Process for Solving the Irreducible Case of Cardan's Method. By Otis H. Kendall, pp. 285-287. V. An Extension of Taylor's Theorem. By J. C. Glashan, pp. 287-288.

Contents of Vol. I, No. 4.

Théorie des Fonctions Numeriques Simplement. Periodiques. Par Edouard Lucas (*Voir pag. 240 et suiv.*), Sections XXIV, XXV, XXVI, XXVII, XXVIII, XXIX, XXX, pp. 289-321; On the two General Reciprocal Methods in Graphical Statics. By Henry T. Eddy, pp. 322-335, Plates IV and V follow page 322; Demonstration of a Fundamental Theorem obtained by Mr. Sylvester. By B. Lipschitz, pp. 336-340; Note on the Theorem contained in Professor Lipschitz's

Paper. By J. J. Sylvester, pp. 341-343; Letter from Mr. Muir to Professor Sylvester on the Word Continuant, p. 344; Extract from a Letter of Dr. Frankland to Mr. Sylvester, pp. 345-349; Applications of Grassmann's Extensive Algebra. By Professor Clifford, pp. 350-358; The Motion of a Point upon the Surface of an Ellipsoid. By Thomas Craig, pp. 359-364; On a Problem of Isomerism, By F. Franklin, pp. 365-368; Note on Indeterminate Exponential Forms. By F. Franklin, pp. 368-369; A Synoptical Table of the Irreducible Invariants and Covariants to a Binary Quintic with a Scholium on a Theorem in Conditional Hyperdeterminants. By J. J. Sylvester, pp. 370-378; The Tangent to the Parabola. By M. L. Holman and E. A. Engler, pp. 379-383; Addenda to Mr. Halsled's Paper on the Bibliography of Hyper-Space and Non-Euclidean Geometry, pp. 384-385; Notes.—I. *Link Work for x^2* (*Extract from a Letter of Professor Cayley to Mr. Sylvester*, p. 386; II *Link-work for the Lemniscate*. (*Extract from a letter of Mr. A. W. Phillips*) p. 386; III Euler's Equations of Motion. By James Loudon, p. 387-388. IV *Condition of a Straight Line Touching a Surface*. By James Loudon, p. 388.

Contents of Vol. II, No. 1.

The Pascal Hexagram by Miss Christine Ladd, pp. 1-12; On the Theory of Flexure.* By William H. Burr, pp. 13-45. Plate I follows page 44.* Foot note reads as follows: (As for instance, Eaton Hodgkinson, who, we believe, made accurate determinations in his subject many years before those whose names are above mentioned, having turned his attention to it as early as 1824—Eds.) Note on the First English Euclid. By George Bruce Halsted, pp. 46-48; On the Fundamental Formulae of Dynamics by J. W. Gibbs, pp. 49-64; Addenda to Bibliography of Hyper-Space and Non-Euclidean Geometry. By George Bruce Halstead. (This journal, Vol. I, pp. 261-276, 384, 385.) Titles marked* were furnished by J. C. Glashan, Ottawa, Canada. Page 61, fourth line from bottom, *before* Geometry, insert appendix to page 61 fourth line from bottom, *after* Thompson, *insert* Fourth Edition, pp. 65-70; Calculation of the minimum N. G. F. of the Binary Seventhic. By Professor Cayley, pp. 71-84; On the Lateral Deviation of Spherical Projectiles. By Henry T. Eddy, pp. 85-88; Note on Determinants and Duadic Disynthemes. By J. J. Sylvester, pp. 89-96; Desiderata and Suggestions, By Professor Cayley. No. 3. The Newton-Fourier Imaginary Problem. On the Complete System of the "Grundformen" of the Binary Quantic of the Ninth Order. By J. J. Sylvester. Extract of a Letter from Sig. A. de Gasparis to Mr. Sylvester; p. 99. Clifford Testimonial Fund, a Fund to be raised for the benefit of Professor Clifford's widow and children, pp. 99-100.

Contents of Vol. II, No. 2.

An Essay on the Calculus of Enlargement. By Emory McClintock, pp. 101-161; The Motion of a Solid in a Fluid. By Thomas Craig, pp. 162-177. Sur l'Analyses indéterminée du troisième degré.—Demonstration de plusieurs théorèmes de M. Sylvester. Par Eduoard Lucas, pp. 178-185. Desiderata and Suggestions. By Professor Cayley. No. 4. Mechanical Construction of Conformable Figures. Notes.—I. Note on Partitions. By F. Franklin, pp. 187-188. II. Some General Formulæ for Integrals of Irrational Functions. By W. I. Stringham, pp. 188-189. III. Note to the Article "On the Theory of Flexure", at page 13, (Vol. II) of this Journal. By William H. Burr, p. 191. IV. Generalization of Leibnitz's Theorem in Statics. Extract of a letter from Professor Crofton to Professor Sylvester, pp. 191-192.

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The contents of these two volumes of this Journal will give the reader a very limited impression of its entire contents. We shall conclude the review of the *American Journal of Mathematics* by incorporating the subjects of the contents of No. 2, Vol. XXXVIII.

Contents of Vol. XXXVIII, No. 2.

On the Classification and Invariantive Characterization of Nilpotent Algebra.* By O. C. Hazlett, pp. 109-138; Determination of the Order of the Groups of Isomorphisms of the Groups of Order p , where p is a prime. By Ross W. Warrott, pp. 139-154; Correspondences Determined by the Bi-tangents of a Quantic. By J. R. Conner, pp. 155-176; Infinite Groups Generated by Conformal Transformations of Period Two (Involutions and Symmetries). By Edward Kasner, pp. 177-184; On the Solutions of Linear Homogenous Difference Equations. By R. D. Carmichael, pp. 185-220.

Early English Arithmetics

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At the time of the invention of printing, Arithmetic was considered both an art and a science. Accordingly there were two types of Arithmeticians, the Algoristic and the Boetian. The former were concerned primarily with calculation, while the latter were interested in the properties of the subject. Tonstoll and Record* were the preservers of the Algoristic method in England, and before the end of the 16th century the ordinary style of commercial Arithmetic, which has prevailed ever since, was in course of establishment. From the time of Robert Record, England was always conspicuous in numerical skill as applied to money. De Morgan says, "Nothing could arise to alter my conviction that the efforts which were made in this country towards the completion of the logarithmic tables in the 17th century were the results of that superiority in calculation."

By some historians, Robert Record is given credit for having written the first text on the subject of Arithmetic in the English language, and they all agree that this was the first text to receive general use in England. In Ball's *History of Mathematics* in Cambridge, there appears this statement: "This text was the earliest English scientific work of any value, and is the best treatise on Arithmetic produced in the 16th century."

Robert Record was born at Tenby, England about 1510. He was educated at Oxford, and later taught Mathematics at Cambridge. The fact that he and Tonstoll, the two outstanding mathematicians of that century, both taught at Cambridge would indicate that this university was becoming an important school of mathematics even at that early date. It is said of him concerning his teaching that †"he rendered clear to all capacities to an extent wholly unprecedented." He was also a doctor of medicine, and later in life became court physician to the King and Queen of England.

The *Ground of Artes*, as Record named his Arithmetic, made its first appearance about 1540. This, as well as other texts which he wrote, is written in the dialogue method, as will appear in the description to follow. The fact that the last known edition appeared in 1673

*Sometimes spelled Recorde.
†Athenæ Cantabrigienses.

is a testimonial to the hold that it had upon the schools of England. It even found its way across the waters, and was used in the American colonies along with Dilworth, Cocker, and other English Arithmetics.

In his text Record made extensive use of the "rule of false assumption," which consists in assuming any number for the unknown quantity, and if on one trial it does not satisfy the given conditions, then it is corrected by the use of proportion as then used in the rule of three. To illustrate, suppose a man spends $\frac{1}{5}$ of his money for groceries, $\frac{1}{5}$ for rent and has \$70 left. How much money had he? Let \$300 represent the original amount. He would then have \$140 left. Then using the proportion $70:140::x:300$, the actual amount is found to be \$150. As the number of types of problems to which this process may be applied is rather limited, it is not as useful as it might appear. Record preferred this method because he could obtain the true result by using the chance answer of "such children or idiots as happen to be in the place." Credit is given to the author for being the first to present the present day method of extracting square root, as well as the first to use two parallel lines to represent equality, which he selected "because than two parallel straight lines no two things can be more equal."

The oldest edition to be found in the British Museum Library was printed in 1543. That this was not the first edition of Record's Arithmetic is indicated in the preface in which is stated that the edition contains "divers new additions." As was the customs of the times much space is used in properly dedicating the book. Eight pages are devoted to "The Most Mighty Prince, Edward the VI, by the Grace of God, King of England, France and Ireland," in which he establishes the fact "undoubtedly as Man is one of the greatest miracles that God ever wrought, so a wise man is plainly the greatest. And therefore was it that some did account the Head of a man the greatest miracle in the world, because not only by the strange workmanship that is in it, but much more because of Reason, Wit, Memory and Imagination, and such other Powers and Works of the mind." He then discusses the opinion of philosophers concerning man and his powers, followed by a statement of the influence of science upon virtue, thus: "As these Sciences did increase so did Virtue increase thereby. But as these Sciences did decay so Virtue lost her estimation and consequently was of little use." Because of the high esteem given to Arithmetic by Robert Record, he concludes his preface to His Majesty in this manner: "How can they either make good laws or maintain them that lack the true knowledge whereby to judge them? And happy may that Realm be accounted where the Prince himself is studi-

ous of Learning. My lowly request to your Majesty is, that this amongst others of my books may pass under the protection of your Highness, whom I beseech God most earnestly and daily to advance in all Honor ahd Princely Regality, and to increase in all Knowledge, Justice, and Godly Policy."

Six pages are then devoted "To the Loving Readers." "Oft times have I lamented with myself the unfortunate condition of England, seeing so many great Clerks to arise in sundry other parts of the World, and so few to appear in this our Nation; whereas for frequency of natural wit (I think) few nations do excell Englishmen. But I cannot impute the cause to any other thing than to the contempt or disregard of Learning. For as Englishmen are inferior to no men in Wit, so they pass all men in vain Pleasures, to which they may attain with great pain and labor." He then uses three pages to deplore the the conditions of the time with respect to learning, concluding as follows: "Therefore, Gentle Reader, though this book can be of small aid to the Learned sort, yet unto the simplest ignorant it may be a good furtherance and means unto knowledge. I will say no more, but let every man judge as he shall see cause. And thus for this time, I will stay my Pen, committing you all to that true Fountain of perfect Number, which wrought the whole World by Number and Measure: He is Trinity and Unity and Glory, Amen."

The table of contents is divided into two parts, the first part being confined to operations and problems involving integers, while the second part is devoted to fractions. One of the outstanding features of the first part is the chapter on Computation by Counters. The entire table follows, in which the first and second "Dialogues" constitute the material devoted to integers.

The contents of the first Dialogue:

1. The declaration of the profit of Arithmetic.
2. Numeration with easie and Large Table.
3. Four fundamental operations.
4. Reduction with diverse declarations of Coynes, Weights and Measures of sundry forms, newly added, with a new Table, containing most part of the gold Coin throughout Christendome, with the true weight and valuation of them now in current English money.
5. Progressions both Arithmetical and Geometrical, with diverse sundry questions touching the same.
6. The Golden Rule, or Rule of Proportion, called the Rule of Three direct.
7. The Golden Rule, or Rule of Three, backward or reverse.
8. The Double Rule, or proportion, direct.
9. The Rule of Proportion composed of five numbers.
10. The Backer Rule, or the second part of the Rule of Proportion, compound.
11. The Rule of Fellowship without time limited.
12. The Rule of Fellowship with time limited.

The Second Dialogue containeth:

12. The *first kinds of Arithmetic wrought by counters.
13. The common kinds of casting by Counters, after the Merchants fashion and Auditors also.

Contents of Second Part:

14. Numeration in Fractions.
15. The order of working Fractions.
16. Reduction of divers Fractions into one denomination in three varieties.
17. Reduction of Fractions of Fractions.
18. Reduction of Improper Fractions.
19. Reduction of Fractions to the smallest Denominator, with easy rules how to convert them thereto.
20. Reduction of a Fraction, and how it may be turned into another fraction, or into what Denomination you list.
21. Addition of Fractions.
22. Subtraction of Fractions.
23. Multiplication of Fractions.
24. Duplication of Fractions.
25. Division of Fractions.
26. Mediation of Fractions.
27. The Golden Rule Direct in Fractions.
28. The Backer, or Reverse Rule in Fractions.
29. The Statute of Assise of Bread and Ale recognized, and applied to the time, with new tables thereunto annexed.
30. The Statute of measuring Ground, with a table thereof faithfully calculated and corrected.
31. The Rule of fellowship, or society, with the reasons of the rules and proofs of their work.
32. To find three numbers in any proportion.
33. The Rule of Allegation, with diverse questions, and the proofs of their works, with many varieties of such solutions.
34. The Rule of Falsehood, or false Position, with diverse questions, and their proofs.

In the text are found many tables to "aid the Reader," named as follows:

1. A large table of Numeration.
2. A table of Multiplication.
3. A table of all gold Coyns in this Realm, with the most usual gold Coyns throughout Christendome, with their several weights of pence and grains, and what they are worth in current money English.
4. The Valuation of Coyns this present year.
5. Certain Tables or notes of the contents of Ale, Beer, Wine, Butter, Soap, Salmon, Eele, etc. both what such vessels ought to contain by the Statute, and what such vessels empty ought to weigh.
6. Table of the quantity of dry Measures, as Pecks, Bushels, Quarters, Weyns, etc.
7. Table of the proportion of measures, touching length and breadth, to-wit, from the inch to the foot, and so to the yard, the Ell, with their parts, the Perch, the Rod, the Furlong, the Mile, etc.
8. Table of Progression Arithmetical which containeth a double Table of Multiplication.

*This refers to Numeration and the four fundamental operations.

9. Table of Demonstration of a figure or measure for the perfect understanding of a fraction of Fractions.
10. Table of the contents of the Statute for the assize of the weight of Bread, from one shilling the quarter to twenty shillings faithfully corrected.
11. Two large tables containing the assize of Bread from three shillings the quarter of wheat, to forty shillings six pence.
12. A necessary Table of the Statute of measuring ground, upon the breadth given, what length it ought to contain.

At the very beginning of the text the Author states that "it were very good to have some understanding and Knowledge of these Figures and notes:

I	1	one	XX	20	Twenty
II	2	two	XL	40	Forty
III	3	three	L	50	Fifty
IV	4	four	LX	60	Sixty
V	5	five	LXX	70	Seventy
VI	6	six	XC	90	Ninety
VII	7	seven	C	100	A hundred
VIII	8	eight	CC	200	Two hundred
IX	9	nine	D	500	Five hundred
X	10	ten	DC	600	Six hundred
XI	11	eleven	M	1000	A thousand
XII	12	twelve	MD	1500	A thousand Five hundred

The dialogue method is used almost exclusively by Record "Because I judge that to be the easiest way of instruction, when the Scholar may ask every doubt orderly, and the Master may answer to his question plainly." The first twenty-four pages of the text are devoted to the declaration of the "profit" of the subject, in which the Scholar seems rather skeptical at first, but finally he is persuaded, and he is made to say to the Master, "Sir, such is your authority in mine estimation, that I am content to consent to your saying, and receive it as truth, though I see none reason that doth lead me there unto." Then the conversation continues by introducing Numbering and its advantages.

Master: "Numbering is so common that no man can do anything alone, and much less bargain with others; he shall still have to do with numbers; this proveth number not to be contemptible and vile, but rather excellent and of high reputation. It is the ground of all men's

affairs, in that without it no tale can be told, no bargaining can be ended, and there are other reasons innumerable which declare number to exceed all praise. Wherefore in all great works are clarks so much versed? Wherefore are auditors so richly paid? What causeth Geometricians to be so greatly enhanced? Because that by number such things they find which otherwise would far excell man's mind."

Scholar: Verily, Sir, if it be so that these men by numbering their cunning do attain, then I see well I was much deceived, and numbering is a more cunning thing than I took it to be.

Master: If numbering were so vile a thing as you did esteem it, then need it not be used in men's communication. Exclude number and answer me this question. How many years old are you?

Scholar: Mumm.

Master: How many days in a week? How many weeks in a year? What lands hath your father? How long is it since you came from him to me?

Scholar: Mumm.

Master: So that if Number want, you answer all by *Mumms*. Thus you may see, what rule Number beareth, and that if number be lacking, it maketh men dumb, so that most questions they must answer Mumm.

Scholar: This will I remember, but how many things are to be learned to attain the Art fully?

Master: There are reckoned commonly seven parts of it. Numeration, Addition, Subtraction, Multiplication, Division, Progressions, and extraction of roots. To these some men add duplation, triplation, and mediation. But as for these they are contained in the other seven, for duplation and triplation are contained under multiplication, and mediation under division.

Scholar: Now I desire you to instruct me in the use of them.

Master: So I will.

Scholar: Even as you please. Then to begin, Numeration is the first in order, what shall I do with it?

Master: First you must know what the thing is, and then after learn the use of the same.

Definitions of the seven operations are stated in the text, independent of the dialogue plan. Numeration is the first to be defined as follows: "Numeration is the Arithmetical skill whereby we may duly value, express and read any number or sum propounded; or else in apt Figures and Places set down any number known or named. Then follow twelve pages of dialogue in which the Master attempts to

make clear this subject. The latter part of the dialogue seems to indicate that the Master has succeeded.

Scholar: Why, then me thinks you put a difference between the Value and the Figures.

Master: So I do. For the Value is one thing and the figures are another, and that cometh by the diversity of Figures, but chiefly in the places wherein they be set.

Scholar: Then must I know here these things, the Value, the Figure, and the Place?

Master: Even so. But yet add order to them as a fourth.

During the process of developing the topic of Numeration the student is taught how to write and read numbers, as well as a complete knowledge of the various Kinds of Numbers. Concerning zero, the Master states "Of the ten figures used in arithmetic one doth signify nothing, which is made like the letter O, and is called a cypher."

In response to the question by the Scholar, Now is there anything else to be learned in Numeration, or have I learned it fully, the Master calls attention to the fact that there are three classes of numbers, Digits, Articles and Mixt. "A digit is a number under 10, as these: 1, 2, 3, 4, 5, 6, 7, 8, 9. And 10 with all others that may be divided into ten parts, nothing remaining, are called articles, such as 10, 20 100, 200..... And that number is called Mixt, that containeth articles, or at the least one article, and a Digit, as 12, 38, 107,....."

In reading numbers the question of beginning at the right or at the left seems to have caused the scholar some trouble, so he finally asks "Why do men reckon the order of the places backward, from right to the left"? To which the Master replies: "In that thing all men do agree, that the Chaldees, which first invented this art did set these figures as they set all their letters, for they write backward as you term it."

In teaching addition, after a discussion of the process, Record makes the Scholar to remark, "That is very easy to do. There came through Cheapside two droves of Cattel: in the first was 848 sheep, and in the second was 186 other Beasts. Those two sums I must write as you taught me. Then if I put the two first Figures together, they make 14. This must I write under 6 and 8 thus:

$$\begin{array}{r}
 848 \\
 186 \\
 \hline
 14
 \end{array}$$

Master. Not so. Here you are twice deceived. First, in going about to add together two sums of sundry things which you ought not to do except to seek only the total number of things, and care not for

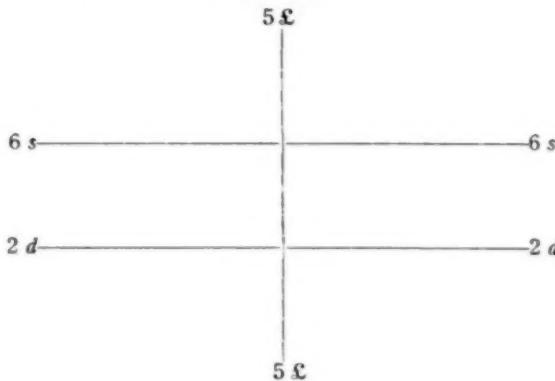
the things; and the other point was in writing 14, which came of 6 and 8, under 6 and 8, which is impossible: for how can two figures of two places be written under one figure and one place?"

To make clear the method of adding numbers of "diverse Denominations" the master makes use of a problem in money. "One man oweth me 22£ 6s. 8d., another owes me 5£ 16s. 6d., and another oweth me 4£ 3s. 0d. I would know what this is all together. Therefore must I set down my great sum, and then the others, everyone under his denomination, agreeing to the greatest sum as here you see, with a line under them:

£	s	d
22	6	8
5	16	6
4	3	0
<hr/>		
32	6	2

During the process of the explanation the Master calls attention to the fact that "I must begin with the right hand where the smallest denomination is, and add them together; and if the sum will make one or more of the next denomination then must I keep it in mind till I come to that place." Step by step the process is explained, and the answer above indicated is obtained.

Casting out the nines is used to prove the process. A vertical line is drawn to represent the highest denomination represented in the problem, here the vertical represents pounds. Then a horizontal is drawn to represent each of the other denominations. The figure represents the process for the problem just used.



The sum of the numbers in the pence column is 14, or 1 shilling and 2 pence. The two is less than 9, so it is placed in its position at the right end of the lower line. Then the sum in the shillings column is 26,

which makes 2 pounds and 6 shillings. Since the 6 is less than 9, it is placed at the right end of the upper horizontal. But there is a total of 32 pounds from the pounds column, which leaves 5 after casting out the nines. This is placed at the top of the vertical column. Now examine the answer from the standpoint of casting out the nines, and results at end of the respective lines are the same, thus proving the accuracy of the work.

"Subtracting, or rebating, is nothing else but an act to withdraw, or abate, one sum from another that the remainder may appear." Simple problems seem to be understood readily by the Scholar, but when asked to subtract 5,278,473 from 8,250,003,456, some difficulty arises for the scholar remarks, "Then I take 7 out of 5, but that I can not do, what shall I do?"

Master: Mark well what I shall tell you, now, how you shall do in this case. In any figure if the nether sum be greater than the figure of the sum that is over him then must you put 10 to the over figure, and then consider how much it is, and out of that whole sum withdraw the nether figure. But now must you mark another thing also that when you do put ten to any figure, you must add one still to the figure of the place that followeth in the nether line. So here we have the whole theory of subtraction. But before we go to multiplication I would instruct you how to examine your work in subtraction whether it be well done or not. Draw under the lowest number a line, and then add this Remainder and all the other that you did subtract before, together, and write the result under the lower line: and if the sum that cometh thereof be equal to the highest of the subtraction then is the subtraction well wrought.

"Multiplication is an operation whereby 2 sums produce the third which third sum shall so many times contain the first, as there are unities in the second. And it serves instead of so many additions. As for example: When I would know how many are 30 times 48, if I should add 48 thirty times, it would be long work. Therefore was multiplication devised."

Scholar: I perceive the commodity of it partly, but I shall not see the full profit of it till I know the full use of it. Therefore, Sir, I beseech you teach me the working of it.

Master: Because the multiplication of great sums can not be performed but by multiplication of digits, therefore I think it best to show you the way of multiplying them. As for the digits under 5 it were but folly to teach any rule. But for the multiplication of greater Digits, thus shall you do: First, set your digits one right under the other, then from the uppermost downwards and the nethermost upward

draw lines so that they make a cross, commonly called St. Andrews Cross, as you see here.

$$\begin{array}{c} 8 \\ \times \\ 7 \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array}$$

Then look to see how much each of them lacketh of 10, and place these differences on the cross as indicated

$$\begin{array}{c} 8 \\ \times \\ 7 \end{array} \quad \begin{array}{c} \diagup 2 \\ \diagdown 3 \end{array} \quad .$$

Then do I draw a line under them,

$$\begin{array}{c} 8 \\ \times \\ 7 \end{array} \quad \begin{array}{c} \diagup 2 \\ \diagdown 3 \end{array}$$

Last of all I multiply the two differences, saying 2 times 3 makes 6, that must I set under the differences. Then must I take one of the differences from the other Digit (not its own) as the lines of the cross warns me. That difference I must write under the digits. And that must be the product of 7×8 .

$$\begin{array}{c} 8 \\ \times \\ 7 \end{array} \quad \begin{array}{c} \diagup 2 \\ \diagdown 3 \end{array} \quad \underline{56}$$

If it is desired to multiply 264 by 29, the two numbers are arranged as we would arrange them today, but the process is quite different. First multiply 9 by 4, and set 36 as indicated. The multiply 9 by 6, and place the 4 directly under the 3 and the 5 at the left of 3. Then the product of 9 and 2 is treated in the same manner, and 8 being placed under the 5, and the 1 at the left. The various products obtained by multiplying by 2 are arranged according to this plan, and the columns are then added.

Record also used a rectangular arrangement for determining the products obtained by use of the various digits in the multiplicand. Using 2036×23 as an example, the following represents this process:

2	0	3	6	
4	0	6	2	2
6	0	9	8	3

This rectangular array is then applied to the problem, making the process appear thus:

$$\begin{array}{r}
 2036 \\
 23 \\
 \hline
 18 \\
 609 \\
 12 \\
 406 \\
 \hline
 46,828
 \end{array}$$

The various partial products are taken directly from the figure. Also it might be noted that the sum of the digits appearing in the various diagonals constitute the digits found in the answer. There is no evidence that Record recognized this fact.

Casting out the nines is used to test the accuracy of the work. After showing the Scholar this method for testing multiplication, the Master suggests "This is a common proof, but the most certain proof is by division, of which I will now instruct you. But now for both your ease and surity I will show you here a table, whereby shall appear the multiplication of all the digits."

1	1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18	
3	9	12	15	18	21	24	27		
4	16	20	24	28	32	36			
5	25	30	35	40	45				
6	36	42	48	54					
7	49	56	63						
8	64	72							
9	81								

The definition of division seemed quite confusing to the Scholar as well it might. For according to the Master, "Division is a distributing of a greater sum by the units of a lesser. Or Division is an arithmetical producing of a third Number, in respect to two propounded numbers, which third Number shall so often contain a unit, as the greater of the two propounded Numbers can contain the lesser." So that "as multiplication does serve in place of many additions, so division may be used in place of many subtractions." The mechanical

process of performing the division is very unwieldly as is illustrated by the following example:

Master: I would divide 365 by 28. Then set I those two sums thus:

$$\begin{array}{r} 365 \\ 28 \end{array}$$

Now I look how many times I may find 2 (which is the last figure of the Divisor) in 3, (which is the last of the number to be divided), and considering that I can take 2 out of 3 but once, I make a crooked line at the right hand of the numbers and within it I set the 1, and that is called the quotient number. Then because that when 2 is taken out of 3, there remaineth 1, I must write that 1 over 3 then cancel the 3 and the 2, then will the figure stand thus

$$\begin{array}{r} 1 \\ 365(1. \\ 28 \end{array}$$

Then come I to the next figure of the divisor, and take it so many times out of the figures that be over it, and look what doth remain, that I must write over them and cancel them as in this example. Therefore I will now take once 8 out of 16, and there remaineth 8, which I must set under the 6, and cross out the 16, and the 8 of the divisor; and then will the figures stand thus:

$$\begin{array}{r} 18 \\ 365(1. \\ 28 \end{array}$$

So I have thus wrought once. When you have thus wrought then must you begin again and write your divisor anew, nearer toward the right hand by one place, as in this example you shall set 2 under 8 and 8 under the 5, thus:

$$\begin{array}{r} 18 \\ 365(1. \\ 288 \\ 2 \end{array}$$

Then as before seek how many times you may take your Divisor out of the number over *him* now.

Scholar: That may I do here 4 times.

Master: True it is that you may find 2 four times in 8; but then mark whether you can find the figure following so many times in the other that is over him. Can you find 8 four times in 5?

Scholar: No, neither yet once.

Master: Therefore take 2 out of 8 once less. Well, then 3 times 2 makes 6. If I take 6 out of 8, there remaineth 2; which 2 with 5 following makes 25, in which sum I find 8 times three also, and therefore I take 3 as the true quotient and write it in the crooked line *before* the 1, thus:

$$\begin{array}{r}
 6 \\
 181 \\
 365(13. \\
 288 \\
 2
 \end{array}$$

By this process it has been discovered that when 365 is divided by 28 the quotient is 13 and the remainder is 1. It might be observed that Record has used this example to illustrate how many months there are in a year, provided there were exactly 28 days in each month. Following this illustrative example it is suggested that 365 be divided by 52, evidently having something of the same idea in mind.

Difficulties arise which are skillfully explained to the Scholar. To illustrate the necessity of placing a zero in the quotient when division is impossible and also showing when the first digit of the divisor is not placed under the first digit of the dividend, the following dialogue takes place:

Master: Now if you perceive the order of division then do you divide 136280 by 452.

Scholar: First I set down the number that should be divided; then do I set the Divisor under the last figure of the over number. Then will it be thus

$$\begin{array}{r}
 136280 \\
 452
 \end{array}$$

Master: Can you take 4 out of 1?

Scholar: I had forgotten, I must set the Divisor one place more forward to the right and the result will be written

$$\begin{array}{r}
 136280. \\
 452
 \end{array}$$

Now I must look how often I may find the last figure of the Divisor (that is 4) in 13, which I may do 3 times." Using the method of division and keeping in mind that operations are performed from left to right the following result is obtained:

$$\begin{array}{r}
 116 \\
 136280(3. \\
 252
 \end{array}$$

Now when the digits of the divisor are set forward so that the 4 stands under the 6, it will be observed that there is no digit directly above 4 that is not crossed off.

The Master makes the observation, "But you may see that over the 4 there is no figure, therefore you must set the divisor forward by another place, and mark because there is no number over it, you shall write in the quotient a cypher."

Twenty-six pages of the text are devoted to the mysteries of division, but evidently this effort on the part of the Master was justified as the scholar finally remarks, "Truely, Sir, these excellent conclusions do wonderfully move me and more make me in love with the art."

Following a thorough discussion of "The Method and Profit" of each of the fundamental operations, Record proceeds to apply these to Reductions, Progressions, Golden Rule and Fellowship. "Reduction is by which all summes of grosse Denominations may be turned into summes of more subtle Denominations. And contrariwise, all summes of subtile Denominations may be brought to summes of groser Denominations." The definition confuses the Scholar, and causes him to ask "What call you grosse Denominations, and subtile Denominations?" To which the Master responds "I call a grosse Denomination which doeth contain under it many other subtler or smaller: as a pound (in respect to shillings) is a grosse Denomination, for it is greater than shillings, and containeth many of them. And shillings (in comparison to pounds) are a subtile Denomination, for because they are lesser than pounds, and so likewise of other things." Then follow instructions concerning operations involved in combining and reducing denominative numbers. These are applied to "Coynes, Weights, and Measures", each of the three being preceded by "certain instructions incident thereto."

"First I have this table wherein is comprehended, not only our current and common coynes, but also the most part of the useful coynes of Christiendom: A table of the names and now valuations of the most usuall Gold Coynes with their several weight of pence and Grams, and what they are worth of current English money this present year, 1640."

Two other tables concerning money appear in the text. "A table of forain Gold Coyne, according to their ancient valuation and severall weight in Pence and Grams", and "A table giving the price of Gold which the bringers in of forrain Gold shall receive at the mint, according to the Kings Majesties Proclamation dated the 14 of May, Anno 1612." Tables concerning weights and measures were then given,

THE NAMES AND TITLES OF THE GOLDE	THE WEIGHT IN		THE VALUE IN	
	PENCE	GRAINES	SHIL.	PENCE
Great Soverain.....	10	0	33	0
Double Sov. K. H.....	8	1	22	0
Double Sov. of Q. E.....	7	7	22	0
Royall.....	4	23	16	6
Half Royall.....	2	1d.	8	3
Old Noble.....	4	6	14	8
Half Noble.....	2	3	7	4
Angell.....	3	8	11	0
Half Angell.....	1	16	5	6
Salute.....	2	5	6	11 ob.
2 parts of Salute.....	1	11	4	7
George Noble.....	3	0	9	9 ob.
Half George Noble.....	1	11	4	11 ob.
First Crown K. H.....	2	9	6	11 ob.
False Crown K. H.....	2	0	5	6
Sover. K. H. best.....	2	14	11	8 ob.q.
Soverain K. H.....	4	0	11	0
Edward Sover.....	3	15d.	11	0
Elizabeth Sover.....	3	15d.	11	0
Elizabeth Crown.....	1	9	5	6
Half Crown.....	0	19	2	9
Unit.....	0	12	22	0
Double Crown.....	3	6	11	0
Britain Crown.....	1	1	5	6
Thistle Crown.....	1	7	4	4 ob.q.
Half Crown.....	0	19d.	2	9
Crosse Dagger.....	3	6d.	11	0
Half Crosse Dagger.....	1	15	5	6
Rose Royall.....	0	21	33	0
Spur Royall.....	4	10d.	16	0
The Angell.....	2	23d.	11	0
Half Angell.....	1	11d.	5	6

after which the Master remarks, "Here will I make an end of Reduction for this time, which though it be counted no kinde of Arithmetic, you see it is no less needful to be known or easier to be done. But in as much as you understand so much as we have entreated of, I will now instruct in Progressions of which two are sufficient for you to be exercised in, of which one I call Arithmetical, and the other Geometrical."

After stating the definition for each, followed by a few illustrations, the Master asks "And do you not also perceive that the whole table of Multiplication may be made from progression Arithmetical if you will begin at the first number of any of them on the left hand, and so proceed right, or any number of the upper and go directly downward?"

The method for finding the sum of n terms of an arithmetical progression is identical with the method now used, but with a slightly different interpretation according as n is odd or even. "If of any such

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

progression you would speadily know the summe, first tell how many numbers there are (which numbers we call places or parcels) and if they be odde, then write the summe down by itself; then adde together the first number and the last, and take half and multiply the result by the number, and the summe that amounteth is the summe of all of those figures added together." If, however, the number of terms is even then the rule states that $\frac{1}{2}$ of the number of terms shall be multiplied by the sum of the first and last terms. "But if you will take one Rule for these both, doe thus: multiply the half of one by the other whole, and the summe will amount all one."

Geometrical progressions are treated in the same manner. The methods of procedure are written in detail, and many examples are discussed involving both types of progressions, indicating clearly the author's plan of attack. The "profit" of the plan is clearly shown in the following example: "If I sold unto you a horse having four shoes, and in every shoe six nayls, with this condition, that you shall pay for the first nayl one bob, for the second nayl two bob, for the third nayl foure bob, and so forth, doubling until the end of all the nayls. Now I ask you how much would the price of the horse come unto?" The Master insists that the 24 terms be found separately, and the sum determined before the Scholar is permitted to state that the result might have been found by resorting to the "Rule of summing up the progression, where I consider that the increase of this summe proceedeth by the multiplication of 2, and therefore multiplying the last summe by 2 also and it yieldeth 16777216, from which I abate the first number, which is 1, and there resteth 16777215, which I would divide by one less than I did multiply: but seeing that this is 1, I need not to divide by it, for 1

(as you have before said) doeth neither multiply nor divide, therefore I do take the number 16777215 as the whole summe of the halfpence." To which the Master replies "That is well done, but I thinke you will buy no horse of that price."

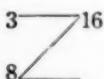
These rules for summing progressions show clearly the advantage of a formula, which, in some cases would replace paragraphs in text books written in the time of Robert Record.

The Golden Rule, or Rule of Proportion direct, called the Rule of Three is introduced by the Master in this manner. "By order of the science there should follow next the *Extraction of Roots of Numbers, which because it is somewhat hard for you yet, I will let it passe for a while, and will teach you the feat of the Rule of Proportion, which for his excellency is called the Golden Rule. Whose use is, by three numbers known to find out any other unknown which you desire to know, as thus. If you pay for your board for three months sixteen shillings, how much shall you pay for eight months?"

"To know this and all such like questions, you shall consider which two of your numbers be of one Denomination, and set those two, one over the other so that the undermost be it that the question is of: as in my question 3 and 8 be both of one Denomination, for they both be months; and because 8 is the number that the question is asked of, I set the 8 undermost, with such a crooked draft of lines.



Then do I set the other number, which is 16, against the 3 at the right side of the line thus:



And now to know my question this must I do: I must multiply the lowermost on the left side by that on the right side and the sum that amounteth I must divide by the highest on the left side: or in planer words thus, I shall multiply the number of which the question is asked (which is called the third number) by a number of another denomination (which is called the second number) the sum that amounteth must I divide by the summe of like denomination (which is called the first). Then for the knowledge of this question I must multiply 8 into 16, and there amounteth 128, which I divide by 3, and it yieldeth 42 shillings, and 2 shillings remaineth which I turn into pence, and there

*This process did not appear in the early editions of the *Ground of Arts*. The 1557 edition of the *Whetstone of Witte* contains an elaborate discussion.

be 24 pence, of which the third part is 8 pence, so the third part of 128 shillings is 42 shillings and 8 pence, which sum I write at the right hand of 8 thus."

$$\begin{array}{r} 3 \diagup \quad 16 \\ 8 \diagup \quad 42s \ 8d \end{array}$$

Several other examples illustrating the Golden Rule direct are found in the text after which the "Golden Rule of Proportion Backward or Reverse" is introduced. In this rule, as Record points out, the order is contrary to the Golden Rule direct in that "The greater the third summe is above the first, the lesser the fourth summe is beneath the second; and this Rule therefore may be called the Backer or Reverse Rule." Some examples are discussed at length, in which the "crooked draught of lines" is used as in the case of the rule direct. The first "Dialogue" closes with an elaborate application of the Rule of Three to fellowship, both with and without time limit.

Some historians suggest that the second dialogue did not appear in the first edition of Record's *Ground of Artes*. *As the oldest edition to which I had access contained this chapter on "Arithmetic wrought by Counters," I have not been able to prove or disprove this statement. On the title page of this early edition there appears a statement concerning "certain additions and corrections." All such changes were made by Record himself. It is, therefore, certain that this dialogue was written by him.—(*To be concluded in February issue.*)

*1543, British Museum Library.

Problem Department

Edited by

ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SOLUTIONS

Late Solution: No. 409 by *Walter B. Clarke*

No. 345. Proposed by *Daniel Arany*, Budapest, Hungary.

Given two coplanar triangles $A_1A_2A_3$ and $A_4A_5A_6$, and the line p in the plane of the two triangles. Let $A'_1, A'_2, A'_3, A'_4, A'_5, A'_6$ be the points where the lines $A_2A_3, A_3A_1, A_1A_2, A_5A_6, A_6A_4$, and A_4A_5 , respectively, meet p . Let A_{41}, A_{42} , and A_{43} be the respective intersections of A_1A_4' and $A_1'A_4$; A_2A_4' and $A_2'A_4$; A_3A_4' and $A_3'A_4$. Prove:

- (1) That the points $A_1, A_2, A_3, A_4, A_{41}, A_{42}, A_{43}$ lie on a conic.
- (2) Two new conics may be formed by substituting the pairs A_5 and A'_5 ; A_6 and A'_6 for A_4 and A'_4 in the set of (1). Show that these three conics meet in a point P .
- (3) Give a straightedge construction for P .

Solution by the *Proposer*.

If we choose the triangle $A_1A_2A_3$ for triangle of reference, the coordinates of the points A_1, A_2, A_3 are

$$h_1, 0, 0 \quad 0, h_2, 0 \quad 0, 0, h_3$$

respectively. Let $x_i, y_i, z_i, i=4,5,6$, be the coordinates of the points A_i , and let the coordinates of the line p be λ_0, μ_0, ν_0 . (Thus the equation satisfied by the coordinates of all points of p is

$$a\lambda_0x + b\mu_0y + c\nu_0z = 0.*$$

*For convenience, the system of line coordinates is usually chosen such that $a=b=c=1$.

Since the equation of the side A_2A_3 is $x=0$ the coordinates of A_1' , the point of intersection of these lines, are found to be

$$\rho x' = 0, \quad \rho y' = cv_0, \quad \rho z' = -b\mu_0.$$

The coordinates of the side A_5A_6 of the triangle $A_4A_5A_6$ are

$$\rho a\lambda_4 = y_5z_6 - y_6z_5 = X_4$$

$$\rho b\mu_4 = z_5x_6 - z_6x_5 = Y_4$$

$$\rho cv_4 = x_5y_6 - x_6y_5 = Z_4$$

where X_4 , Y_4 , Z_4 are the cofactors of x_4 , y_5 , z_6 in the determinant

$$|x_4y_5z_6|.$$

Therefore the coordinates of A_4' are

$$\rho x_4' = cv_0Y_4 - b\mu_0Z_4$$

$$\rho y_4' = a\lambda_0Z_4 - cv_0X_4$$

$$\rho z_4' = b\mu_0X_4 - a\lambda_0Y_4.$$

Now the coordinates of the line $A_1'A_4$ are

$$\rho a\lambda_{41} = y_1'z_4 - y_4z_1' = -cv_0z_4 - b\mu_0y_4,$$

$$\rho b\mu_{41} = z_1'x_4 - z_4x_1' = b\mu_0x_4,$$

$$\rho cv_{41} = x_1'y_4 - x_4y_1' = cv_0x_4;$$

and the coordinates of the line A_1A_4' are

$$\rho a\lambda_{14} = y_1z_4' - y_4z_1 = 0,$$

$$\rho b\mu_{14} = z_1x_4' - z_4x_1 = -z_4',$$

$$\rho cv_{14} = x_1'y_4' - x_4'y_1 = y_4';$$

so that the coordinates of their intersection, A_{41} , are

$$\rho x_{41} = b\mu_0x_4y_4' + cv_0x_4z_4',$$

$$\rho y_{41} = (cv_0z_4 + b\mu_0y_4)y_4',$$

$$\rho z_{41} = (cv_0z_4 + b\mu_0y_4)z_4'.$$

The equation of every conic passing through the vertices of the triangle of reference is

$$a_1yz + b_1zx + c_1xy = 0.$$

If a_1, b_1, c_1 are determined from the equations

$$a_1 y_4 z_4 + b_1 z_4 x_4 + c_1 x_4 y_4 = 0$$

$$a_1 y_{41} z_{41} + b_1 z_{41} x_{41} + c_1 x_{41} y_{41} = 0,$$

the conic will pass through A_4 and A_{41} also. Thus we have

$$\rho a_1 = x_4 x_{41} (y_{41} z_4 - y_4 z_{41}),$$

$$\rho b_1 = y_4 y_{41} (z_{41} x_4 - z_4 x_{41}),$$

$$\rho c_1 = z_4 z_{41} (x_{41} y_4 - x_4 y_{41}).$$

Replacing x_{41}, y_{41}, z_{41} by their values and noting that the parentheses are the coordinates of $A_4 A_{41}$, or what is the same, those of $A_4 A_1'$, we find

$$\rho a_1 = (b \mu_0 y_4 + c v_0 z_4) (-b \mu_0 y_4' - c v_0 z_4') x_4 x_4,$$

$$\rho b_1 = (b \mu_0 y_4 + c v_0 z_4) b \mu_0 y_4' x_4 y_4,$$

$$\rho c_1 = (b \mu_0 y_4 + c v_0 z_4) c v_0 z_4' x_4 z_4,$$

but since A_4' lies on p , whence

$$-b \mu_0 y_4' - c v_0 z_4' = a \lambda_0 x_4',$$

we get $\rho a_1 = a \lambda_0 x_4 x_4'$, $\rho b_1 = b \mu_0 y_4 y_4'$, $\rho c_1 = c v_0 z_4 z_4'$.

From the form of this result, it is evident that we would have found it again by using A_{42} or A_{43} instead of A_{41} . Thus the truth of (1) is demonstrated.

When the values of the coordinates of A_4' are substituted, a_1, b_1, c_1 may be put in another form,

$$\rho a_1 = a \lambda_0 x_4 (c v_0 Y_4 - b \mu_0 Z_4),$$

$$\rho b_1 = b \mu_0 y_4 (a \lambda_0 Z_4 - c v_0 X_4),$$

$$\rho c_1 = c v_0 z_4 (b \mu_0 X_4 - a \lambda_0 Y_4).$$

The analogous coefficients of the equations of the conics passing through the points A_5 and A_{51} ; A_6 and A_{61} are

$$\rho a_2 = a \lambda_0 x_5 (c v_0 Y_5 - b \mu_0 Z_5),$$

$$\rho b_2 = b \mu_0 y_5 (a \lambda_0 Z_5 - c v_0 X_5),$$

$$\rho c_2 = c v_0 z_5 (b \mu_0 X_5 - a \lambda_0 Y_5);$$

$$\rho a_3 = a \lambda_0 x_6 (c v_0 Y_6 - b \mu_0 Z_6),$$

$$\rho b_3 = b \mu_0 y_6 (a \lambda_0 Z_6 - c v_0 X_6),$$

$$\rho c_3 = c v_0 z_6 (b \mu_0 X_6 - a \lambda_0 Y_6).$$

where $X_5, Y_5, Z_5, X_6, Y_6, Z_6$ are the remaining cofactors of the determinant $|x_5y_5z_5|$. From the last 9 equations it appears that

$$a_1 + a_2 + a_3 = 0, \quad b_1 + b_2 + b_3 = 0, \quad c_1 + c_2 + c_3 = 0,$$

whence the determinant $|a_1b_2c_3| = 0$, signifying that the three conics, passing through the vertices of the triangle of reference, meet in a common fourth point.

Under the correlation,

$$x = 1/\lambda, \quad y = 1/\mu, \quad z = 1/v,$$

the three conics correspond to the collinear points

$$a_1\lambda + b_1\mu + c_1v = 0,$$

$$a_2\lambda + b_2\mu + c_2v = 0,$$

$$a_3\lambda + b_3\mu + c_3v = 0.$$

The pole of the line determined by these points, with respect to the triangle of reference, is thus the fourth point in which the three conics meet.

No. 397. Proposed by *Paul D. Thomas*, Southeastern State College, Oklahoma.

Find the envelope of all spheres which are bisected by two given intersecting spheres.

Solution by the *Proposer*.

If a sphere is to be bisected by two given spheres its center must have equal powers with respect to both spheres and hence must lie in their radical plane. Take the radical plane of the two given spheres as $x = 0$, and their line of centers as the line $y = z = 0$. Then if r is the radius of the circle of intersection of the two given spheres, the family of spheres bisected by the two given spheres may be written

$$F = x^2 + (y - v)^2 + (z - t)^2 - r^2 + (v^2 + t^2) = 0,$$

where $(0, r, t)$ is any point in the radical plane. The elimination of v and t among $F = 0, F_v = 0, F_t = 0$ produces the envelope $2x^2 + y^2 + z^2 = 2r^2$, which is an ellipsoid of revolution, the generating ellipse being $2x^2 + y^2 = 2r^2$.

No. 398. Proposed by *John H. Giese*, New Brunswick, N. J.

An elliptic billiard table is assumed to be frictionless and to have perfectly elastic cushions. It has one pocket, at a vertex. Show that,

if a ball placed at a focus be struck in any direction, it is sure to drop into the pocket.

Solution by the *Proposer*.

Let F_1 and F_2 be the foci, e the eccentricity, and P_0 an arbitrary point on the ellipse. Let $P_n (n=0,1,2,\dots)$ be a sequence of points of the ellipse such that $P_{2i-2}P_{2i-1}$ passes through F_2 and $P_{2i-1}P_{2i}$ passes through F_1 . By the focal property of the ellipse F_1P_0 and the successive segments P_nP_{n+1} evidently correspond to the path of the ball. If P_0 is a vertex, the ball rebounds at once into a pocket; hence in what follows P_0 is not a vertex.

Let θ_{2i} be the angles (between 0 and π) formed by the rays F_1F_2 and F_2P_{2i} , and θ_{2i+1} that formed by F_1F_2 and F_1P_{2i+1} . Since θ_n is an exterior angle of a triangle in which θ_{n+1} is an opposite interior angle we have $\pi > \theta_n > \theta_{n+1} > 0$. Thus the bounded monotonic sequence $\{\theta_n\}$ has a limit, which we may designate by φ .

Now introduce two polar coordinate systems, each with its pole at a focus and its initial ray passing through the other focus. (If the polar angle which now corresponds to θ_0 happens to be negative, a reflection of the ellipse in its major axis will make θ_0 positive. Since P_n and P_{n+1} are on opposite sides of the major axis, every θ_n is now positive as in the previous paragraph.) The equation of the ellipse in either coordinate system becomes

$$r = em / (1 - e \cos \theta),$$

where m is a constant.

Suppose F is the focus on P_nP_{n+1} and F' the focus on $P_{n+1}P_{n+2}$. Let h_n and h_{n+1} be the numerical values of the projections of FP_n and $F'P_{n+1}$ on the minor axis. Then

$$h_n = |FP_n| \sin \theta_n, \quad h_{n+1} = |FP_{n+1}| \cdot |\sin(\theta_n + \pi)|.$$

If we put

$$(1) \quad R(\theta) = (1 - e \cos \theta) / (1 + e \cos \theta)$$

we obtain

$$\frac{h_{n+1}}{h_n} = \frac{FP_{n+1}}{FP_n} = R(\theta_n) \quad \text{and}$$

$$(2) \quad h_{n+1} = \frac{h_{n+1}}{h_n} \cdot \frac{h_n}{h_{n-1}} \cdots \frac{h_1}{h_0} \cdot h_0 = h_0 \prod_{s=0}^n R(\theta_s).$$

We have above $\lim \theta_n = \varphi$. Then $0 \leq \varphi < \pi$. First suppose $\varphi < \pi/2$. Given any ψ such that $\varphi < \psi < \pi/2$, there exists an N such that $\varphi < \theta_n < \psi$ for every $n > N$. By (1) $R(\theta_n) < R(\psi)$ for $n > N$ and by (2),

$$(3) \quad h_{n+1} < h_0 \left[\prod_{s=0}^N R(\theta_s) \right] \cdot [R(\psi)]^{n-N}.$$

Since $R(\psi) < 1$, we have

$$\lim_{n \rightarrow \infty} h_n = 0.$$

$\varphi > \pi/2$ is impossible. For, with $\theta_n > \varphi > \pi/2$, we would have $R(\theta_n) > R(\varphi)$ from which by (2) we have

$$h_{n+1} > h_0 [R(\varphi)]^{n+1}$$

which ultimately exceeds the semi-minor axis. To show that $\varphi = \pi/2$ is impossible, let B_1 and B_2 be the ends of the latera recta through F_1 and F_2 corresponding to $\theta = \pi/2$. Let B_1F_2 and B_2F_1 intersect the ellipse again at C_1 and C_2 . In terms of polar coordinates at F_2 , C_1 has a coordinate θ such that $\pi/2 < \theta < \pi$. Suppose $\lim \theta_n = \pi/2$. There are then infinitely many points P_n closer to B_2 than C_1 . For such a P_n , the line $P_nF_2P_{n+1}$ determines P_{n+1} on the minor arc B_1C_2 for which $\theta_{n+1} < \pi/2$. Since $\{\theta_n\}$ is a decreasing sequence with $\lim \theta_n = \pi/2$, we have a contradiction. Thus always $\lim h_n = 0$, and the path of the billiard ball ultimately follows arbitrarily closely the major axis with the pocket at one end.

No. 423. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

In triangle ABC , side AB is fixed in size and position and angle C is constant. Determine the locus of the feet of the internal and external bisectors of angles A and B .

Solution by *David L. Bowman*, student, Colgate University.

For reference, select the pole at A and the polar axis in the direction A to B . Let P , the foot of the internal bisector of the angle A on the side BC , have the coordinates (r, θ) . From the triangle APB we have the relation

$$r/\sin B = c/\sin(A/2 + C).$$

Expressing the angles in terms of C and θ there results the equation for the locus of the point P

$$(1) \quad r = c \cdot \sin(2\theta + C)/\sin(\theta + C).$$

In a similar manner it will be found that the locus of the point P' , the foot of the external bisector of the angle A on the side BC extended, is also given by the equation (1). The locus of the point P is

given by equation (1) as Θ varies from 0 to $\pi/2 - C/2$. The locus of the point P' is given by the same equation, but Θ varies from $\pi/2$ to $\pi - C/2$. The remaining portions of the complete locus do not apply to the problem as stated.

It is evident that the feet of the internal and external bisectors of angle B will lie on a curve which is symmetrical to the locus (1) with respect to the perpendicular bisector of the line AB . The equation of this locus, if referred to the same pole and axis, is

$$(2) \quad r = \cos \frac{1}{2}(\Theta + C) / \cos \frac{1}{2}(\Theta - C).$$

This locus is completely traced as the angle Θ changes from 0 to 2π , but the parts which apply to this problem are given as Θ varies from 0 to $\pi - C$ and from π to 2π , respectively.

PROPOSALS

No. 442. Proposed by *Paul D. Thomas*, Southeastern State College, Oklahoma.

Find the equation of the geodesics on a surface whose linear element is given by $ds^2 = (u^2 - v^2)(v^2 du^2 + u^2 dv^2) / u^2 v^2$ and show that if points on the surface correspond to points in the plane $z=0$ through $u=a \cdot \sec(\log x)$, $v=1/y$, the geodesics correspond to the central conics $x^2 - 2abxy + b^2 = 0$, where a and b are constants.

No. 443. Proposed by *Walter B. Clarke*, San Jose, California.

The isosceles triangle ABC with $AC = BC$ has orthocenter H . Let P be any point on the circle through A , B , and H , and let AP cut BC in A' , BP cut AC in B' . Show that $AA' = BB'$.

No. 444. Proposed by *H. T. R. Aude*, Colgate University.

In the scale of three, each of the fractions which have a certain integer N for denominator requires three digits in the repetend. In the scale of six, the fractions having a second integer M for denominator require six digits in the repetend. The numbers N and M when written in the denary scale use the same digits. Find M and N .

No. 445. Proposed by *W. V. Parker*, Louisiana State University.

The circumcenter of triangle $A_i B_i C_i$ is within the triangle. If perpendiculars drawn from the circumcenter to the sides of the triangle are extended to meet the circumcircle in A_{i+1} , B_{i+1} , C_{i+1} , show

that as i takes on the values $0, 1, 2, \dots$ the triangle approaches an equilateral triangle.

No. 446. Proposed by *R. F. Rinehart*, Case School of Applied Science, Cleveland, Ohio.

Perform the following integration:

$$\int \frac{\sin x \, dx}{e^x + \cos x + \sin x}$$

No. 447. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Inscribe a triangle ABC in a given circle O so that the projections of B and C on the bisector of angle A and the projection of A on BC are the vertices of a triangle of maximum area.

No. 448. Proposed by *M. S. Robertson*, Rutgers University.

A long rectangular strip of paper has one corner folded over so as just to touch the opposite side. Show that the creases formed in this way envelope an arc of a parabola whose directrix and latus rectum produced are the two long sides of the rectangular strip.

No. 449. Proposed by *Paul D. Thomas*, Southeastern State College, Oklahoma.

A variable circle is tangent to one of two perpendicular lines and intercepts a chord of given length on the other. Find the locus of the center.

No. 450. Proposed by *Alice M. H'Doubler*, student, Bryn Mawr College.

A certain youth was asked his age
By one who seemed to be a sage;
To whom the youth made this reply,
Sir, if you wish your skill to try,
Eight times my age increased by four,
A perfect square, nor less nor more;
Its triple square plus nine must be
Another square as you will see.
He tried but sure it posed him quite,
His answer being far from right.
You skilled in Science I implore
This mystic number to explore.

(From *The Scientific Journal*, 1818, Question 13, Proposed by *M. O'Shannessy*, Teacher of Mathematics, Albany.)

Bibliography and Reviews

Edited by
H. A. SIMMONS and JOHN W. CELL

Mathematics—Its Magic and Mastery. By Aron Bakst. D. Van Nostrand & Co., New York, 1941. xiv + 790 pages. \$3.95.

It may be heresy, but mathematics even to the mathematician sometimes seems just a little dull and humdrum. This is probably more true in the case of the college instructor whose professional labors are mainly confined to the teaching of freshman and sophomore courses. As the years go by, his work becomes more or less routine, and unless he makes a special effort his lectures are apt to become more and more uninteresting to his students as he puts less and less time in their preparation.

All this is preliminary to a suggestion that every teacher of mathematics should have at his command as many illustrations from every walk of life as possible. *Mathematics—Its Magic and Mastery*, although not written with this as its primary purpose, is a book admirably suited for reinforcing lectures in algebra, trigonometry, and analytic geometry.

This book was designed to make mathematics interesting to the layman. To this end it is written in a most refreshingly informal style, and for the most part rigorous proofs are avoided. In spite of its informal style, the book presents a thorough picture of the fundamentals of arithmetic, algebra, trigonometry, and analytics. These topics are literally developed by applying them to many diverse problems associated with, for example, rapid calculations, cryptography, installment buying, chain-letters, insurance, mechanics, and ballistics.

The prospective reader who has even a grain of curiosity must necessarily investigate a chapter whose heading in the Table of Contents is: The Firing Squad and Mathematics.—What Happens When You Pull the Trigger?—How Strong Is a Bullet?—You May Not Hit the Target, but You'll Get a Kick Out of This.—The Path of a Bullet.—Big Bertha's Secret.—Aerial Artillery: Bigger than Bertha.—The Algebra of a Fired Shell.—Bertha's Shell and Johnny's Top.—Out of the Firing Pan into the Fire.—

Mathematics—Its Magic and Mastery, although written in unorthodox style and in the language of the layman, is a book any mathematician will enjoy. It would be fun to teach a one year course with this book as text.

North Carolina State College.

H. M. NAHICKIAN.

Tables prepared as the Project for the Computation of Mathematical Tables, conducted by the Works Projects Administration for the City of New York, under the Sponsorship of the National Bureau of Standards of Washington, D. C.:

MT1. *Tables of the First Ten Powers of the Integers from 1 to 1000.* 1939. viii + 80 pages. \$0.50.

MT2. *Tables of the Exponential Function e^x .* 1939. xv + 535 pages. \$2.00.

This collection contains tables for e^x with x ranging from 0 to 1 with $\Delta x = 0.0001$ and correct to 18 decimals; from 1 to 2.5 correct to 15 decimals with $\Delta x = 0.001$; and

from 5 to 10 correct to 12 decimals with $\Delta x = 0.01$. The tables also include e^{-x} from 0 to 2.5 correct to 18 decimals with $\Delta x = 0.0001$.

MT3. *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.* 1939. xvii + 405 pages. \$2.00.

These tables range from 0 to 2 correct to 9 decimals with $\Delta x = 0.0001$, and from 0 to 10 correct to 9 decimals with $\Delta x = 0.1$.

MT4. *Tables of Sines and Cosines for Radian Arguments.* 1940. xxix + 275 pages. \$2.00.

These tables for $\sin x$ and $\cos x$ range for x from 0 to 25 with $\Delta x = 0.001$ and correct to 8 decimals, and from 0 to 100 correct to 8 decimals with $\Delta x = 1$.

MT5. *Tables of Sine, Cosine, and Exponential Integrals.* Vol. I. 1940. xxvi + 444 pages. \$2.00.

This volume includes tables for $Si(x)$, $Ci(x)$, $Ei(x)$, and $-Ei(-x)$ for x from 0 to 2 correct to 9 decimals and $\Delta x = 0.0001$.

MT6. *Tables of Sine, Cosine, and Exponential Integrals.* Vol. II. 1940. xxxvii + 225 pages. \$2.00.

This volume includes tables for $Si(x)$, $Ci(x)$, and $Ei(x)$ for x from 0 to 10 with $\Delta x = 0.001$ and correct to 10 decimals, and for $-Ei(-x)$ from 0 to 10 with $\Delta x = 0.001$ and correct to 9 decimals.

MT7. *Tables of Natural Logarithms.* Vol. I. 1941. xvii + 501 pages. \$2.00.

This volume contains the logarithms correct to 15 decimals of all the integers from 1 to 50,000.

In January, 1938, Dr. Lyman J. Briggs of the Bureau of Standards sponsored a meeting to discuss and outline numerical tabulations for important functions. There were present: from the "Committee on Bibliography of Mathematical Tables and Aids to Computation" of the National Research Council: Professors A. A. Bennett, R. C. Archibald, H. T. Davis, and D. H. Lehmer; representing the Works Progress Administration: Mr. M. Morrow, Mr. C. B. Lawrence, Jr., and Dr. A. N. Lowan; Dr. C. E. Van Orstrand, Mr. E. C. Crittenton, and Dr. L. B. Tuckerman. Since then additions have been made to the scope of the project as the result of consultation with outstanding men of science both in this country and abroad.

These volumes have all been printed by the photo off-set method and are quite readable. Each volume contains necessary supplementary tables to be used with the main tables. In each volume is an introduction describing the nature of the tables, methods of interpolation to be used and the accuracy therewith obtainable, and the various checks and multiple checks that have been applied to the final manuscript forms to rid them of all possible errors.

The construction of these sets of tables is an American counterpart of the work of the Tables Committee of the British Association for the Advancement of Science and there is little duplication in results. Those directly or indirectly responsible for these and subsequent projects of similar character have performed a very valuable service.

North Carolina State College.

JOHN W. CELL.

Early Military Books in the University of Michigan Libraries. By Thomas M. Spaulding and Louis C. Karpinski. The University of Michigan Press, Ann Arbor, 1941. xvi+45 pages +photographs of 37 plates. Price \$2.00.

This little book contains an appreciative *Foreword*, written by President Alexander G. Ruthven, of the University of Michigan.

The *Introduction* to the book, written by the authors, Colonel Spaulding and Professor Karpinski, is thoroughly explanatory of the nature and purpose of the book.

This book is primarily a list of references to articles on "military science and art" which appeared in the period from 1493 to 1800. There are 372 of these references, which seem to have been well selected. Each one of them is to one or more of 31 libraries in the United States. To each library there is given a code name: C, L, T, SS stand, respectively, for (University of Michigan libraries) Clements Library of American History, Law Library, Transportation Library, Stephen Spaulding Collection (General Library).—and these libraries alone contain considerably more than half of the references. The abbreviations for the other 27 libraries are the ones that are employed by the Union Catalogue of the Library of Congress. These 27 include such famous ones as those of Harvard, Yale, Princeton, Brown (the John Hay), the University of Chicago, and the John Crerar of Chicago.

Three and one-half pages are devoted to a *General Index*; and a single page, to an "Index of military books by mathematicians and of mathematical works with sections on military science". These indexes appear on pages 41 to 45 of the book, and they should enable the reader to find quickly the works of a particular author and also to find readily the mathematical references of the book. Many of the references are non-mathematical.

The 37 plates which constitute the last part of the book are interesting for their variety,—they are written on a wide range of subjects and in many languages—and novel appearances.

We recommend the book to anyone who is apt to desire to peruse, or read carefully, any considerable amount of the literature in question; and we believe that the book should be in all libraries.

Northwestern University.

H. A. SIMMONS.

College Algebra. Second edition. By H. P. Pettit and P. Luteyn. John Wiley and Sons, Inc., New York, 1941. xiv+247 pages. \$1.90.

The book presents college algebra from a new and interesting standpoint. Instead of starting with the conventional treatment of equations (theory of equations, remainder and factor theorems, etc.), the authors begin with explanations of notations, such as functional notation, summation notation, and the many varieties of notations customarily used in studying and using algebraic quantities and forms. While preparing the student for the future chapters, this method offers good opportunity for an incidental review of the preparatory elementary algebra without devoting excessive time to review exercises. Then, in Chapter II, the authors proceed with a fundamental task, that of graphical representation of functions. However, instead of following this up immediately with a study of variation and functional behavior, they treat this graphical representation as another example of mathematical notation or language. Then in the subsequent chapters they proceed to further types of notation, such as those in permutations and combinations, probability, binomial theorem, mathematical induction, determinants. The purpose seems to be to impress the student with the

necessity of learning the notation, as constituting the expression of a precise language, and along with this to develop in him habits of careful thinking and some degree of manipulative skill.

Another outstanding feature of the book is the set of chapters on *Problem-Solving*, with preparatory explanations on how to apply algebra to practical problems, including physical and engineering problems. These chapters are more or less evenly distributed throughout the book; they contain a commendable choice of interesting and valuable problems, evidently designed and arranged to develop not merely manipulative skill but also careful, accurate thinking, especially in preparation for applications to engineering problems.

There is an answer-list to the odd-numbered problems, a short table of 4-place logarithms, and a table of the exponentials e^x and e^{-x} .

There are some misprints; on page 223, line 5, in Example 2 on the *Series Ratio Test*, $2n\sqrt{1}$ (occurring in the denominator) should be $2n-1$, and in the preface in line 19 of page viii the pages referred to should be 66-78.

University of Florida.

B. F. DOSTAL.

A First Year of College Mathematics. By Henry J. Miles. John Wiley and Sons, Inc., New York, 1941. xvii+607 pages. \$3.00.

This book constitutes a one-year combination course in algebra, analytic geometry (plane and solid), trigonometry, and calculus. As an example of what it accomplishes in the direction of developing the interrelationships of these subjects, and hence also in reinforcing the student's grasp of them, the proofs for the trigonometric addition theorems are here based on the rotation formulas of analytic geometry. Such a procedure is very valuable for another reason; viz., to prepare students who intend to take more advanced mathematics, such as group theory. There are many such valuable methods employed in this book.

Another outstanding feature of the book is the very large number of exceedingly good "combination" problems and practical (physical, engineering, chemical, biological, business, all with most modern aspects) problems given in review lists, profusely distributed throughout the book. Answers to the odd-numbered problems are given.

Still another outstanding feature is the directness with which the book proceeds to its main tasks from the first chapter to the last. For example, no time is lost in preparatory review material, as its purpose is accomplished incidentally in the working of the regular exercises that are devoted primarily to new topics.

To suggest the very logical arrangement of the chapters, we mention that the first and second chapters are on *Rectangular Coordinates* and *Graphs*, respectively, while the third chapter is on *Functions*.

Unusual logic of procedure is suggested by the fact that the author refers to *direction cosines* of a line in two dimensional space in the same way that he does in three dimensional space. Such consistency of form is laudable; for it eases the student's task.

Worthy of special mention are the chapters on *Empirical Equations* and *Foundations of Algebra*. The former chapter calls for use of logarithmic and semi-logarithmic coordinate paper in problems of curve fitting, which are solved by the method of averages and that of least squares. The latter chapter is invaluable for giving the student an introduction into the methods of modern mathematical logic and an insight into the powerful generalizing and abstracting processes that are being employed at present in mathematical research.

A set of 4-place and 5-place logarithmic, trigonometric and natural logarithmic tables is included, and all of the tables occupy a total of but 10 pages.

The appearance of this book marks a long step forward in the direction of improving the teaching of freshman mathematics.

University of Florida.

B. F. DOSTAL.

Exterior Ballistics, being Chapters X and XII from *Elements of Ordnance*, prepared under the direction of Lt. Col. Thomas H. Jayes, John Wiley and Sons, Inc., New York, 1940. \$1.00.

The appearance of this little book is timely, and it presents to the mathematical reader in somewhat compact form material that he might not find conveniently explained elsewhere. The first chapter, the one which gave the title to the book, is a 72 page discussion of exterior ballistics. The second chapter of 25 pages is devoted to the newer subject of Bombing from Airplanes.

One scarcely ever goes far in the study of mathematics without paying a little attention to the motion of projectiles. Texts on calculus and on mechanics consider the simple case where the resistance of the air can be neglected. This problem is very simple, and the results are satisfactory for projectiles with very low velocity. But the problem that must be solved in order to construct the firing tables necessary for modern artillery, especially anti-aircraft, is a very difficult one. The first chapter of the present book is only a sketch, but it nevertheless touches upon all aspects of the general problem. It is well up to date. The sections devoted to a discussion of the forces that act upon projectiles draw upon the paper of Fowler, Gallop, Lock, and Richmond, which gave a more complete theory than had appeared before, a theory supported quite extensively by experimental results. Thus the differential equations of motion that are set down in the little book under review are more complete than those customarily encountered.

In connection with the computation of trajectories a brief description of the older methods is given, followed by an explanation of numerical integration, as a basis for the modern procedure. This part of the discussion draws much from the work done by Moulton and other American mathematicians during World War I. Topics are touched upon without being developed in full, and the person who wishes to master the subject will find it necessary to consult other works. For this purpose he is furnished with quite an adequate bibliography.

The second chapter particularly presents material that may not be available elsewhere. The subject is one that some teachers of mathematics may find it necessary to study during months ahead. No really advanced mathematics is involved, but a reader will not master the chapter in a short time. A teacher is likely to wonder about the difficulty of presenting the subject to students with inadequate mathematical preparation. Although the reputed excellence of the American bomb sight has been much publicized, laymen generally know little if anything as to what such an instrument must accomplish, if it is to allow the bomber to place his missiles upon the target. The basic principles of the sight are explained, but the various devices that must be added in order to make the necessary computations rapidly and mechanically are naturally only referred to. How intricate an instrument is needed is shown by the statement, "A good bomb sight will therefore include a means for indicating to the pilot the course he should fly." When the sight does this it furnishes information that in all probability could be obtained otherwise only by several changes of course and several calculations.

K. P. WILLIAMS.